Appendix K

Thermodynamic formalism

Being Hungarian is not sufficient. You also must be talented.
— Zsa Zsa Gabor

(G. Vattay)

In the preceding chapters we characterized chaotic systems via global quantities such as averages. It turned out that these are closely related to very fine details of the dynamics like stabilities and time periods of individual periodic orbits. In statistical mechanics a similar duality exists. Macroscopic systems are characterized with thermodynamic quantities (pressure, temperature and chemical potential) which are averages over fine details of the system called microstates. One of the greatest achievements of the theory of dynamical systems was when in the sixties and seventies Bowen, Ruelle and Sinai made the analogy between these two subjects explicit. Later this "Thermodynamic Formalism" of dynamical systems became widely used making it possible to calculate various fractal dimensions. We sketch the main ideas of this theory and show how periodic orbit theory helps to carry out calculations.

K.1 Rényi entropies

As we have already seen trajectories in a dynamical system can be characterized by their symbolic sequences from a generating Markov partition. We can locate the set of starting points $\mathcal{M}_{s_1 s_2 ... s_n}$ of trajectories whose symbol sequence starts with a given set of n symbols $s_1 s_2 ... s_n$. We can associate many different quantities to these sets. There are geometric measures such as the volume $V(s_1 s_2 ... s_n)$, the area $A(s_1 s_2 ... s_n)$ or the length $l(s_1 s_2 ... s_n)$ of this set. Or in general we can have some measure $\mu(\mathcal{M}_{s_1 s_2 ... s_n}) = \mu(s_1 s_2 ... s_n)$ of this set. As we have seen in (22.10) the most important is the natural measure, which is the probability that an ergodic trajectory visits the set $\mu(s_1 s_2 ... s_n) = P(s_1 s_2 ... s_n)$. The natural measure is additive.

Summed up for all possible symbol sequences of length n it gives the measure of the whole state space:

$$\sum_{s_1 s_2 ... s_n} \mu(s_1 s_2 ... s_n) = 1 \tag{K.1}$$

expresses probability conservation. Also, summing up for the last symbol we get the measure of a one step shorter sequence

$$\sum_{s_n} \mu(s_1 s_2 ... s_n) = \mu(s_1 s_2 ... s_{n-1}).$$

As we increase the length (n) of the sequence the measure associated with it decreases typically with an exponential rate. It is then useful to introduce the exponents

$$\lambda(s_1 s_2 ... s_n) = -\frac{1}{n} \log \mu(s_1 s_2 ... s_n). \tag{K.2}$$

To get full information on the distribution of the natural measure in the symbolic space we can study the distribution of exponents. Let the number of symbol sequences of length n with exponents between λ and $\lambda + d\lambda$ be given by $N_n(\lambda) d\lambda$. For large n the number of such sequences increases exponentially. The rate of this exponential growth can be characterized by $g(\lambda)$ such that

$$N_n(\lambda) \sim \exp(n g(\lambda)) .$$

The knowledge of the distribution $N_n(\lambda)$ or its essential part $g(\lambda)$ fully characterizes the microscopic structure of our dynamical system.

As a natural next step we would like to calculate this distribution. However it is very time consuming to calculate the distribution directly by making statistics for millions of symbolic sequences. Instead, we introduce auxiliary quantities which are easier to calculate and to handle. These are called partition sums

$$Z_n(\beta) = \sum_{s_1 s_2 ... s_n} \mu^\beta(s_1 s_2 ... s_n), \tag{K.3}$$

as they are obviously motivated by Gibbs type partition sums of statistical mechanics. The parameter β plays the role of inverse temperature $1/k_B T$ and $E(s_1 s_2 ... s_n) = -\log \mu(s1 s_2 ... s_n)$ is the energy associated with the microstate labeled by $s_1 s_2 ... s_n$ We are tempted also to introduce something analogous with the Free energy. In dynamical systems this is called the Rényi entropy [G.5] defined by the growth rate of the partition sum

$$K_\beta = \lim_{n\to\infty} \frac{1}{n} \frac{1}{1-\beta} \log \left(\sum_{s_1 s_2 ... s_n} \mu^\beta(s_1 s_2 ... s_n) \right). \tag{K.4}$$

In the special case $\beta \to 1$ we get Kolmogorov entropy

$$K_1 = \lim_{n\to\infty} \frac{1}{n} \sum_{s_1 s_2 ... s_n} -\mu(s_1 s_2 ... s_n) \log \mu(s_1 s_2 ... s_n),$$

while for $\beta = 0$ we recover the topological entropy

$$h_{top} = K_0 = \lim_{n\to\infty} \frac{1}{n} \log N(n),$$

where $N(n)$ is the number of existing length n sequences. To connect the partition sums with the distribution of the exponents, we can write them as averages over the exponents

$$Z_n(\beta) = \int d\lambda N_n(\lambda) \exp(-n\lambda\beta),$$

where we used the definition (K.2). For large n we can replace $N_n(\lambda)$ with its asymptotic form

$$Z_n(\beta) \sim \int d\lambda \exp(ng(\lambda)) \exp(-n\lambda\beta).$$

For large n this integral is dominated by contributions from those λ^* which maximize the exponent

$$g(\lambda) - \lambda\beta.$$

The exponent is maximal when the derivative of the exponent vanishes

$$g'(\lambda^*) = \beta. \tag{K.5}$$

From this equation we can determine $\lambda^*(\beta)$. Finally the partition sum is

$$Z_n(\beta) \sim \exp(n[g(\lambda^*(\beta)) - \lambda^*(\beta)\beta]).$$

Using the definition (K.4) we can now connect the Rényi entropies and $g(\lambda)$

$$(\beta - 1)K_\beta = \lambda^*(\beta)\beta - g(\lambda^*(\beta)). \tag{K.6}$$

Equations (K.5) and (K.6) define the Legendre transform of $g(\lambda)$. This equation is analogous with the thermodynamic equation connecting the entropy and the

free energy. As we know from thermodynamics we can invert the Legendre transform. In our case we can express $g(\lambda)$ from the Rényi entropies via the Legendre transformation

$$g(\lambda) = \lambda\beta^*(\lambda) - (\beta^*(\lambda) - 1)K_{\beta^*(\lambda)}, \tag{K.7}$$

where now $\beta^*(\lambda)$ can be determined from

$$\frac{d}{d\beta^*}[(\beta^* - 1)K_{\beta^*}] = \lambda. \tag{K.8}$$

Obviously, if we can determine the Rényi entropies we can recover the distribution of probabilities from (K.7) and (K.8).

The periodic orbit calculation of the Rényi entropies can be carried out by approximating the natural measure corresponding to a symbol sequence by the expression (22.10)

$$\mu(s_1, ..., s_n) \approx \frac{e^{n\gamma}}{|\Lambda_{s_1 s_2 ... s_n}|}. \tag{K.9}$$

The partition sum (K.3) now reads

$$Z_n(\beta) \approx \sum_i \frac{e^{n\beta\gamma}}{|\Lambda_i|^\beta}, \tag{K.10}$$

where the summation goes for periodic orbits of length n. We can define the characteristic function

$$\Omega(z, \beta) = \exp\left(-\sum_n \frac{z^n}{n} Z_n(\beta)\right). \tag{K.11}$$

According to (K.4) for large n the partition sum behaves as

$$Z_n(\beta) \sim e^{-n(\beta-1)K_\beta}. \tag{K.12}$$

Substituting this into (K.11) we can see that the leading zero of the characteristic function is

$$z_0(\beta) = e^{(\beta-1)K_\beta}.$$

On the other hand substituting the periodic orbit approximation (K.10) into (K.11) and introducing prime and repeated periodic orbits as usual we get

$$\Omega(z,\beta) = \exp\left(-\sum_{p,r} \frac{z^{n_p r} e^{\beta\gamma n_p r}}{r|\Lambda_p^r|^{\beta}} \right).$$

We can see that the characteristic function is the same as the zeta function we introduced for Lyapunov exponents (G.12) except we have $ze^{\beta\gamma}$ instead of z. Then we can conclude that the Rényi entropies can be expressed with the pressure function directly as

$$P(\beta) = (\beta - 1)K_\beta + \beta\gamma, \tag{K.13}$$

since the leading zero of the zeta function is the pressure. The Rényi entropies K_β, hence the distribution of the exponents $g(\lambda)$ as well, can be calculated via finding the leading eigenvalue of the operator (G.4).

From (K.13) we can get all the important quantities of the thermodynamic formalism. For $\beta = 0$ we get the topological entropy

$$P(0) = -K_0 = -h_{top}. \tag{K.14}$$

For $\beta = 1$ we get the escape rate

$$P(1) = \gamma. \tag{K.15}$$

Taking the derivative of (K.13) in $\beta = 1$ we get Pesin's formula [G.2] connecting Kolmogorov entropy and the Lyapunov exponent

$$P'(1) = \overline{\lambda} = K_1 + \gamma. \tag{K.16}$$

exercise K.1

It is important to note that, as always, these formulas are strictly valid for nice hyperbolic systems only. At the end of this Chapter we discuss the important problems we are facing in non-hyperbolic cases.

On figure K.2 we show a typical pressure and $g(\lambda)$ curve computed for the two scale tent map of Exercise K.4. We have to mention, that all typical hyperbolic dynamical system produces a similar parabola like curve. Although this is somewhat boring we can interpret it like a sign of a high level of universality: The exponents λ have a sharp distribution around the most probable value. The most probable value is $\lambda = P'(0)$ and $g(\lambda) = h_{top}$ is the topological entropy. The average value in closed systems is where $g(\lambda)$ touches the diagonal: $\overline{\lambda} = g(\overline{\lambda})$ and $1 = g'(\overline{\lambda})$.

Next, we are looking at the distribution of trajectories in real space.

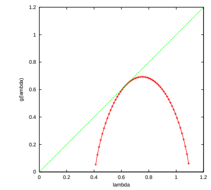


Figure K.1:

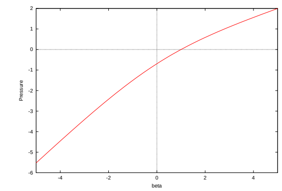


Figure K.2: $g(\lambda)$ and $P(\beta)$ for the map of exercise K.4 at $a = 3$ and $b = 3/2$.

K.2 Fractal dimensions

By looking at the repeller we can recognize an interesting spatial structure. In the 3-disk case the starting points of trajectories not leaving the system after the first bounce form two strips. Then these strips are subdivided into an infinite hierarchy of substrings as we follow trajectories which do not leave the system after more and more bounces. The finer strips are similar to strips on a larger scale. Objects with such self similar properties are called *fractals*.

We can characterize fractals via their local scaling properties. The first step is to draw a uniform grid on the surface of section. We can look at various measures in the square boxes of the grid. The most interesting measure is again the natural measure located in the box. By decreasing the size of the grid ϵ the measure in a given box will decrease. If the distribution of the measure is smooth then we expect that the measure of the ith box is proportional with the dimension of the section

$$\mu_i \sim \epsilon^d.$$

If the measure is distributed on a hairy object like the repeller we can observe unusual scaling behavior of type

$$\mu_i \sim \epsilon^{\alpha_i},$$

where α_i is the local "dimension" or Hölder exponent of the object. As α is not necessarily an integer here we are dealing with objects with fractional dimensions. We can study the distribution of the measure on the surface of section by looking at the distribution of these local exponents. We can define

$$\alpha_i = \frac{\log \mu_i}{\log \epsilon},$$

the local Hölder exponent and then we can count how many of them are between α and $\alpha + d\alpha$. This is $N_\epsilon(\alpha)d\alpha$. Again, in smooth objects this function scales simply with the dimension of the system

$$N_\epsilon(\alpha) \sim \epsilon^{-d},$$

while for hairy objects we expect an α dependent scaling exponent

$$N_\epsilon(\alpha) \sim \epsilon^{-f(\alpha)}.$$

$f(\alpha)$ can be interpreted [G.7] as the dimension of the points on the surface of section with scaling exponent α. We can calculate $f(\alpha)$ with the help of partition sums as we did for $g(\lambda)$ in the previous section. First, we define

$$Z_\epsilon(q) = \sum_i \mu_i^q. \tag{K.17}$$

Then we would like to determine the asymptotic behavior of the partition sum characterized by the $\tau(q)$ exponent

$$Z_\epsilon(q) \sim \epsilon^{-\tau(q)}.$$

The partition sum can be written in terms of the distribution function of α-s

$$Z_\epsilon(q) = \int d\alpha N_\epsilon(\alpha)\epsilon^{q\alpha}.$$

Using the asymptotic form of the distribution we get

$$Z_\epsilon(q) \sim \int d\alpha \epsilon^{q\alpha - f(\alpha)}.$$

As ϵ goes to zero the integral is dominated by the term maximizing the exponent. This α^* can be determined from the equation

$$\frac{d}{d\alpha^*}(q\alpha^* - f(\alpha^*)) = 0,$$

leading to

$$q = f'(\alpha^*).$$

Finally we can read off the scaling exponent of the partition sum

$$\tau(q) = \alpha^* q - f(\alpha^*).$$

In a uniform fractal characterized by a single dimension both α and $f(\alpha)$ collapse to $\alpha = f(\alpha) = D$. The scaling exponent then has the form $\tau(q) = (q-1)D$. In case of non uniform fractals we can introduce generalized dimensions [G.9] D_q via the definition

$$D_q = \tau(q)/(q-1).$$

Some of these dimensions have special names. For $q = 0$ the partition sum (K.17) counts the number of non empty boxes $\bar{N}_\epsilon$. Consequently

$$D_0 = -\lim_{\epsilon\to 0} \frac{\log \bar{N}_\epsilon}{\log \epsilon},$$

is called the box counting dimension. For $q = 1$ the dimension can be determined as the limit of the formulas for $q \to 1$ leading to

$$D_1 = \lim_{\epsilon\to 0} \sum_i \mu_i \log \mu_i / \log \epsilon.$$

This is the scaling exponent of the Shannon information entropy [G.11] of the distribution, hence its name is *information dimension*.

Using equisize grids is impractical in most of the applications. Instead, we can rewrite (K.17) into the more convenient form

$$\sum_i \frac{\mu_i^q}{\epsilon^\tau(q)} \sim 1. \tag{K.18}$$

If we cover the ith branch of the fractal with a grid of size l_i instead of ϵ we can use the relation [K.5]

$$\sum_i \frac{\mu_i^q}{l_i^\tau(q)} \sim 1, \tag{K.19}$$

the non-uniform grid generalization of K.18. Next we show how can we use the periodic orbit formalism to calculate fractal dimensions. We have already seen that the width of the strips of the repeller can be approximated with the stabilities of the periodic orbits placed within them

$$l_i \sim \frac{1}{|\Lambda_i|}.$$

Then using this relation and the periodic orbit expression of the natural measure we can write (K.19) into the form

$$\sum_i \frac{e^{q\gamma n}}{|\Lambda_i|^{q-\tau(q)}} \sim 1, \qquad \text{(K.20)}$$

where the summation goes for periodic orbits of length n. The sum for stabilities can be expressed with the pressure function again

$$\sum_i \frac{1}{|\Lambda_i|^{q-\tau(q)}} \sim e^{-nP(q-\tau(q))},$$

and (K.20) can be written as

$$e^{q\gamma n} e^{-nP(q-\tau(q))} \sim 1,$$

for large n. Finally we get an implicit formula for the dimensions

$$P(q-(q-1)D_q) = q\gamma. \qquad \text{(K.21)}$$

Solving this equation directly gives us the partial dimensions of the multifractal repeller along the stable direction. We can see again that the pressure function alone contains all the relevant information. Setting $q = 0$ in (K.21) we can prove that the zero of the pressure function is the box-counting dimension of the repeller

$$P(D_0) = 0.$$

Taking the derivative of (K.21) in $q = 1$ we get

$$P'(1)(1 - D_1) = \gamma.$$

This way we can express the information dimension with the escape rate and the Lyapunov exponent

$$D_1 = 1 - \gamma/\overline{\lambda}. \qquad \text{(K.22)}$$

If the system is bound ($\gamma = 0$) the information dimension and all other dimensions are $D_q = 1$. Also since $D_1 0$ is positive (K.22) proves that the Lyapunov exponent must be larger than the escape rate $\overline{\lambda} > \gamma$ in general.

exercise K.4
exercise K.5

Résumé

In this chapter we have shown that thermodynamic quantities and various fractal dimensions can be expressed in terms of the pressure function. The pressure function is the leading eigenvalue of the operator which generates the Lyapunov exponent. In the Lyapunov case β is just an auxiliary variable. In thermodynamics it plays an essential role. The good news of the chapter is that the distribution of locally fluctuating exponents should not be computed via making statistics. We can use cyclist formulas for determining the pressure. Then the pressure can be found using short cycles + curvatures. Here the head reaches the tail of the snake. We just argued that the statistics of long trajectories coded in $g(\lambda)$ and $P(\beta)$ can be calculated from short cycles. To use this intimate relation between long and short trajectories effectively is still a research level problem.

Commentary

Remark K.1 Mild phase transition. In non-hyperbolic systems the formulas derived in this chapter should be modified. As we mentioned in 22.1 in non-hyperbolic systems the periodic orbit expression of the measure can be

$$\mu_0 = e^{\gamma n}/|\Lambda_0|^{\delta},$$

where δ can differ from 1. Usually it is 1/2. For sufficiently *negative* β the corresponding term $1/|\Lambda_0|^{\beta}$ can dominate (K.10) while in (K.3) $e^{\gamma n}/|\Lambda_0|^{\delta\beta}$ plays no dominant role. In this case the pressure as a function of β can have a kink at the critical point $\beta = \beta_c$ where $\beta_c \log|\Lambda_0| = (\beta_c - 1)K_{\beta_c} + \beta_c\gamma$. For $\beta < \beta_c$ the pressure and the Rényi entropies differ

$$P(\beta) \neq (\beta - 1)K_\beta + \beta\gamma\,.$$

This phenomena is called phase transition. This is however not a very deep problem. We can fix the relation between pressure and the entropies by replacing $1/|\Lambda_0|$ with $1/|\Lambda_0|^{\delta}$ in (K.10).

Remark K.2 Hard phase transition. The really deep trouble of thermodynamics is caused by intermittency. In that case we have periodic orbits with $|\Lambda_0| \to 1$ as $n \to \infty$. Then for $\beta > 1$ the contribution of these orbits dominate both (K.10) and (K.3). Consequently the partition sum scales as $Z_n(\beta) \to 1$ and both the pressure and the entropies are zero. In this case quantities connected with $\beta \leq 1$ make sense only. These are for example the topological entropy, Kolmogorov entropy, Lyapunov exponent, escape rate, D_0 and D_1. This phase transition cannot be fixed. It is probably fair to say that quantities which depend on this phase transition are only of mathematical interest and not very useful for characterization of realistic dynamical systems.

Exercises

K.1. **Thermodynamics in higher dimensions.** Define Lyapunov exponents as the time averages of the eigenexponents of the Jacobian matrix J

$$\mu^{(k)} = \lim_{t\to\infty} \frac{1}{t} \log |\Lambda_k^t(x_0)|, \tag{K.23}$$

as a generalization of (17.32).

Show that in d dimensions Pesin's formula is

$$K_1 = \sum_{k=1}^{d} \mu^{(k)} - \gamma, \tag{K.24}$$

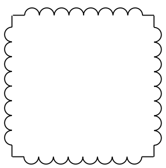
where the summation goes for the positive $\mu^{(k)}$-s only. Hint: Use the d-dimensional generalization of (K.9)

$$\mu_p = e^{n\gamma}/|\prod_k \Lambda_{p,k}|,$$

where the product goes for the expanding eigenvalues of the Jacobian matrix of p-cycle. (G. Vattay)

K.2. **Stadium billiard Kolmogorov entropy.** (continuation of exercise 8.4.) Take $a = 1.6$ and $d = 1$ in the stadium billiard figure 8.1, and estimate the Lyapunov exponent by averaging over a very long trajectory. Biham and Kvale [K.14] estimate the discrete time Lyapunov to $\lambda \approx 1.0 \pm .1$, the continuous time Lyapunov to $\lambda \approx 0.43 \pm .02$, the topological entropy (for their symbolic dynamics) $h \approx 1.15 \pm .03$.

K.3. **Entropy of rugged-edge billiards.** Take a semi-circle of diameter ε and replace the sides of a unit square by $\lfloor 1/\varepsilon \rfloor$ semi-circle arcs.



(a) Is the billiard ergodic as $\varepsilon \to 0$?

(b) (hard) Show that the entropy of the billiard map is

$$K_1 \to -\frac{2}{\pi} \ln \varepsilon + \text{const},$$

as $\varepsilon \to 0$. (Hint: do not write return maps.)

(c) (harder) Show that when the semi-circles of the stadium billiard are far apart, say L, the entropy for the flow decays as

$$K_1 \to \frac{2 \ln L}{\pi L}.$$

K.4. **Two scale map.** Compute all those quantities - dimensions, escape rate, entropies, etc. - for the repeller of the one dimensional map

$$f(x) = \begin{cases} 1 + ax & \text{if} \quad x < 0, \\ 1 - bx & \text{if} \quad x > 0. \end{cases} \tag{K.25}$$

where a and b are larger than 2. Compute the fractal dimension, plot the pressure and compute the $f(\alpha)$ spectrum of singularities.

K.5. **Transfer matrix.** Take the unimodal map $f(x) = \sin(\pi x)$ of the interval $I = [0, 1]$. Calculate the four preimages of the intervals $I_0 = [0, 1/2]$ and $I_1 = [1/2, 1]$. Extrapolate $f(x)$ with piecewise linear functions on these intervals. Find a_1, a_2 and b of the previous exercise. Calculate the pressure function of this linear extrapolation. Work out higher level approximations by linearly extrapolating the map on the 2^n-th preimages of I.

References

[K.1] J. Balatoni and A. Renyi, *Publi. Math. Inst. Hung. Acad.Sci.* **1**, 9 (1956); (english translation **1**, 588 (Akademia Budapest, 1976)).

[K.2] Ya.B. Pesin, *Uspekhi Mat. Nauk* **32,** 55 (1977), [*Russian Math. Surveys* **32,** 55 (1977)].