

## Appendix C

# Finding cycles

(C. Chandre)

### C.1 Newton-Raphson method

#### C.1.1 Contraction rate

CONSIDER A  $d$ -DIMENSIONAL MAP  $x' = f(x)$  with an unstable fixed point  $x_*$ . The Newton-Raphson algorithm is obtained by iterating the following map

$$x' = g(x) = x - (J(x) - \mathbf{1})^{-1} (f(x) - x).$$

The linearization of  $g$  near  $x_*$  leads to

$$x_* + \epsilon' = x_* + \epsilon - (J(x_*) - \mathbf{1})^{-1} (f(x_*) + J(x_*)\epsilon - x_* - \epsilon) + O(\|\epsilon\|^2),$$

where  $\epsilon = x - x_*$ . Therefore,

$$x' - x_* = O((x - x_*)^2).$$

After  $n$  steps and if the initial guess  $x_0$  is close to  $x_*$ , the error decreases super-exponentially

$$g^n(x_0) - x_* = O((x_0 - x_*)^{2^n}).$$

### C.1.2 Computation of the inverse

The Newton-Raphson method for finding  $n$ -cycles of  $d$ -dimensional mappings using the multi-shooting method reduces to the following equation

$$\begin{pmatrix} \mathbf{1} & & & -Df(x_n) \\ -Df(x_1) & \mathbf{1} & & \\ & \cdots & \mathbf{1} & \\ & & -Df(x_{n-1}) & \mathbf{1} \end{pmatrix} \begin{pmatrix} \delta_1 \\ \delta_2 \\ \cdots \\ \delta_n \end{pmatrix} = - \begin{pmatrix} F_1 \\ F_2 \\ \cdots \\ F_n \end{pmatrix}, \quad (\text{C.1})$$

where  $Df(x)$  is the  $[d \times d]$  Jacobian matrix of the map evaluated at the point  $x$ , and  $\delta_m = x'_m - x_m$  and  $F_m = x_m - f(x_{m-1})$  are  $d$ -dimensional vectors. By some straightforward algebra, the vectors  $\delta_m$  are expressed as functions of the vectors  $F_m$ :

$$\delta_m = - \sum_{k=1}^m \beta_{k,m-1} F_k - \beta_{1,m-1} (\mathbf{1} - \beta_{1,n})^{-1} \left( \sum_{k=1}^n \beta_{k,n} F_k \right), \quad (\text{C.2})$$

for  $m = 1, \dots, n$ , where  $\beta_{k,m} = Df(x_m)Df(x_{m-1}) \cdots Df(x_k)$  for  $k < m$  and  $\beta_{k,m} = \mathbf{1}$  for  $k \geq m$ . Therefore, finding  $n$ -cycles by a Newton-Raphson method with multiple shooting requires the inverting of a  $[d \times d]$  matrix  $\mathbf{1} - Df(x_n)Df(x_{n-1}) \cdots Df(x_1)$ .

### C.2 Hybrid Newton-Raphson / relaxation method



Consider a  $d$ -dimensional map  $x' = f(x)$  with an unstable fixed point  $x_*$ . The transformed map is the following one:

$$x' = g(x) = x + \gamma C(f(x) - x),$$

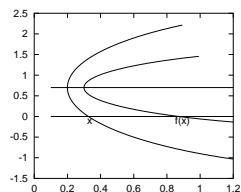
where  $\gamma > 0$  and  $C$  is a  $d \times d$  invertible constant matrix. We note that  $x_*$  is also a fixed point of  $g$ . Consider the stability matrix at the fixed point  $x_*$

$$A_g = \left. \frac{dg}{dx} \right|_{x=x_*} = \mathbf{1} + \gamma C(A_f - \mathbf{1}).$$

The matrix  $C$  is constructed such that the eigenvalues of  $A_g$  are of modulus less than one. Assume that  $A_f$  is diagonalizable: In the basis of diagonalization, the matrix writes:

$$\tilde{A}_g = \mathbf{1} + \gamma \tilde{C}(\tilde{A}_f - \mathbf{1}),$$

**Figure C.1:** Illustration of the optimal Poincaré surface. The original surface  $y = 0$  yields a large distance  $x - f(x)$  for the Newton iteration. A much better choice is  $y = 0.7$ .



where  $\tilde{A}_f$  is diagonal with elements  $\mu_i$ . We restrict the set of matrices  $\tilde{C}$  to diagonal matrices with  $\tilde{C}_{ii} = \epsilon_i$  where  $\epsilon_i = \pm 1$ . Thus  $\tilde{A}_g$  is diagonal with eigenvalues  $\gamma_i = 1 + \gamma\epsilon_i(\mu_i - 1)$ . The choice of  $\gamma$  and  $\epsilon_i$  is such that  $|\gamma_i| < 1$ . It is easy to see that if  $\text{Re}(\mu_i) < 1$  one has to choose  $\epsilon_i = 1$ , and if  $\text{Re}(\mu_i) > 1$ ,  $\epsilon_i = -1$ . If  $\lambda$  is chosen such that

$$0 < \gamma < \min_{i=1, \dots, d} \frac{2|\text{Re}(\mu_i) - 1|}{|\mu_i - 1|^2},$$

all the eigenvalues of  $A_g$  have modulus less than one. The contraction rate at the fixed point for the map  $g$  is then  $\max_i |1 + \gamma\epsilon_i(\mu_i - 1)|$ . If  $\text{Re}(\mu_i) = 1$ , it is not possible to stabilize  $x_*$  by the set of matrices  $\gamma C$ .

From the construction of  $C$ , we see that  $2^d$  choices of matrices are possible. For example, for 2-dimensional systems, these matrices are

$$C \in \left\{ \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} \right\}.$$

For 2-dimensional dissipative maps, the eigenvalues satisfy  $\text{Re}(\mu_1)\text{Re}(\mu_2) \leq \det Df < 1$ . The case  $(\text{Re}(\mu_1) > 1, \text{Re}(\mu_2) > 1)$  which is stabilized by  $\begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$  has to be discarded. The minimal set is reduced to three matrices.

### C.2.1 Newton method with optimal surface of section



(F. Christiansen)

In some systems it might be hard to find a good starting guess for a fixed point, something that could happen if the topology and/or the symbolic dynamics of the flow is not well understood. By changing the Poincaré section one might get a better initial guess in the sense that  $x$  and  $f(x)$  are closer together. In figure C.1 there is an illustration of this. The figure shows a Poincaré section,  $y = 0$ , an initial guess  $x$ , the corresponding  $f(x)$  and pieces of the trajectory near these two points.

If the Newton iteration does not converge for the initial guess  $x$  we might have to work very hard to find a better guess, particularly if this is in a high-dimensional system (high-dimensional might in this context mean a Hamiltonian system with