ChaosBook.org chapter

relativity for cyclists

5 August 2011, version 13.4

Das Problem

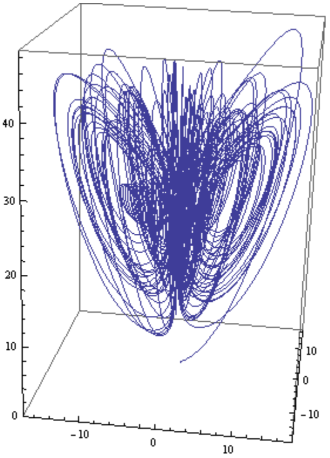
complex Lorenz equations

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{y}_1 \\ \dot{y}_2 \\ \dot{z} \end{bmatrix} = \begin{bmatrix} -\sigma x_1 + \sigma y_1 \\ -\sigma x_2 + \sigma y_2 \\ (\rho_1 - z)x_1 - \rho_2 x_2 - y_1 - e y_2 \\ \rho_2 x_1 + (\rho_1 - z)x_2 + e y_1 - y_2 \\ -bz + x_1 y_1 + x_2 y_2 \end{bmatrix}$$

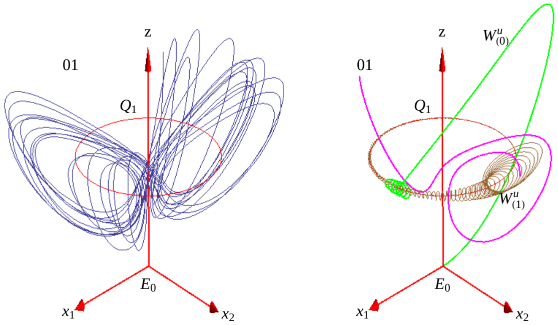
$\rho_1 = 28, \rho_2 = 0, b = 8/3, \sigma = 10, e = 1/10$

- A typical $\{x_1, x_2, z\}$ trajectory
- superimposed: a trajectory whose initial point is close to the relative equilibrium Q_1

attractor



continuous symmetry induces drifts



- generic chaotic trajectory (blue)
- E_0 equilibrium
- E_0 unstable manifold - a cone of such (green)
- Q_1 relative equilibrium (red)
- Q_1 unstable manifold, one for each point on Q_1 (brown)
- relative periodic orbit $\overline{01}$ (purple)

die Lösung

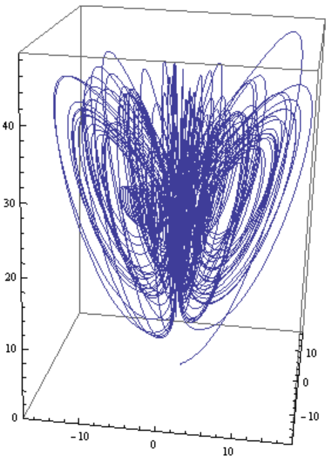
what to do?

it's a mess

the goal

reduce this messy strange attractor to something simple

attractor



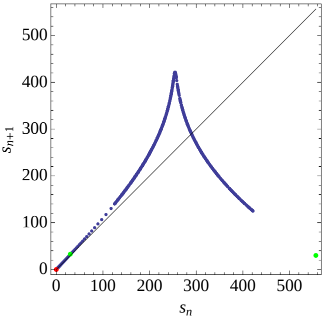
die Lösung

the goal attained

but it will cost you

must learn how to reduce (quotient) the $SO(2)$ symmetry

1D return map!



die Lösung

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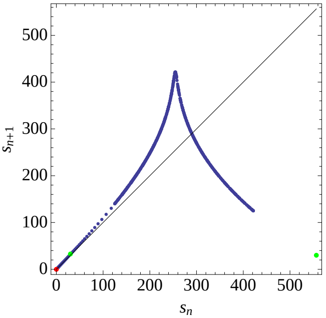


how? hang on, that's what we'll explain here symmetries of dynamics

a flow $\dot{x} = v(x)$ is G-equivariant if

$$v(x) = g^{-1}\, v(g\, x)\,, \qquad \text{for all } g \in G\,.$$

definition: Lie group

a topological group G such that
(1) G has the structure of a smooth differential manifold
(2) composition map $G \times G \to G : (g, h) \to gh^{-1}$ is smooth

mystified?

just think “aha, like the rotation group $SO(3)$...”

example: $\mathrm{SO}(2)$ invariance

complex Lorenz equations equations are invariant under a $\mathrm{SO}(2)$ rotation by finite angle θ:

$$g(\theta) = \begin{pmatrix} \cos\theta & \sin\theta & 0 & 0 & 0 \\ -\sin\theta & \cos\theta & 0 & 0 & 0 \\ 0 & 0 & \cos\theta & \sin\theta & 0 \\ 0 & 0 & -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

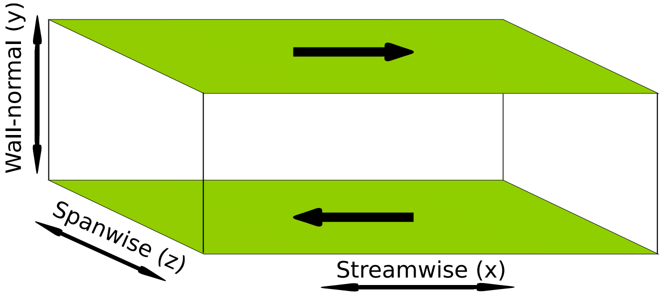
state space decomposition

1. $m = 0$ $\mathrm{SO}(2)$-invariant subspace: z-axis
2. $m = 1$ subspace with multiplicity 2

example: abelian group $SO(2)$
$SO(2)$: rotations in a plane
reflection $(x, y) \to (-x, y)$ excluded ($\det g = -1$)

If the group G actions consists of 2 such rotations which commute, for example a $3D$ box or pipe with two periodic boundary conditions, the group G is an Abelian group that sweeps out a T^2 torus example: symmetries of plane Couette flow

Navier-Stokes flow between two countermoving planes



- box periodic in streamwise and spanwise directions
- eqs. invariant under streamwise, spanwise flips
- $O(2) \times O(2)$ continuous symmetry

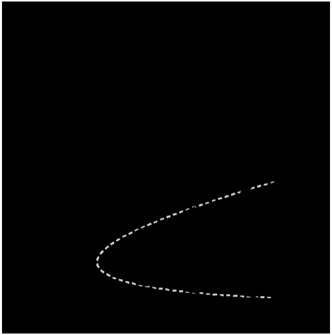
example: continuous symmetries of pipe flow

pipe flow

- periodic streamwise, spanwise
- eqs. under azimuthal flip invariant

a) $SO(2)_z \times O(2)_\theta$ symmetry

b) laminar sol. is invariant



group orbits for any $x \in \mathcal{M}$, the group orbit $\mathcal{M}_x$ of x is the set of all group actions

$$\mathcal{M}_x = \{g\,x \mid g \in G\} \subset \mathcal{M}$$

states in $\mathcal{M}_x$ are physically equivalent example: group orbit of a pipe flow relative equilibrium $\bar{x}' =$

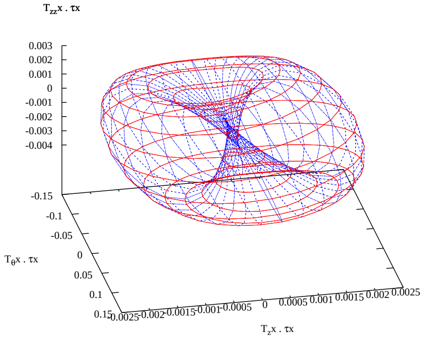
Kerswell *et al* $N2\_M1$ solution, ($Re = 2400$, stubby $L = 2.5D$ pipe)

$\mathrm{SO}(2) \times \mathrm{SO}(2)$ symmetry
$\Rightarrow$ group orbit is 2-torus

projected on

- 2 $\bar{x}'$ group tangents
- 3. axis along the curvature direction

a very smooth, lower branch solution

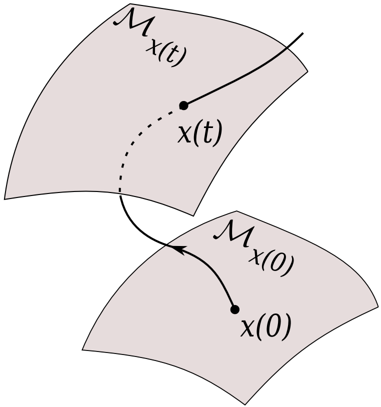


2d group orbit (in 100,000 dimension state space) traced out by

- equal increment translations in θ (dashed blue)
- equal increments in z (solid red)

foliation by group orbits

group orbits

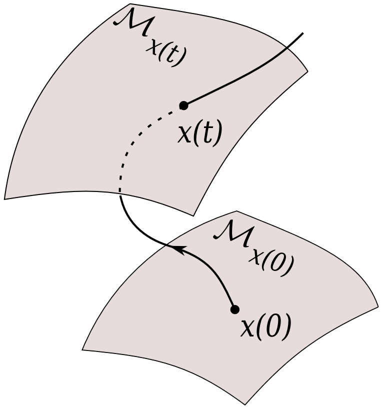


group orbit $\mathcal{M}_x$ of x is the set of all group actions

$$\mathcal{M}_x = \{g\,x \mid g \in G\}$$

foliation by group orbits

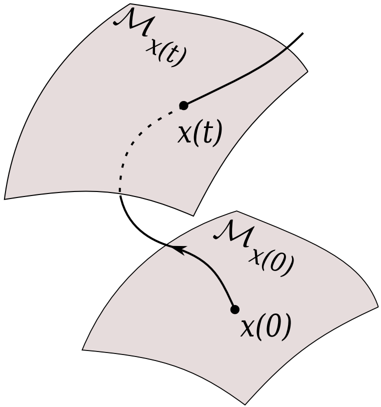
group orbits



any point on the manifold $\mathcal{M}_{x(t)}$ is equivalent to any other

foliation by group orbits

group orbits



action of a symmetry group foliates the state space into a union of group orbits

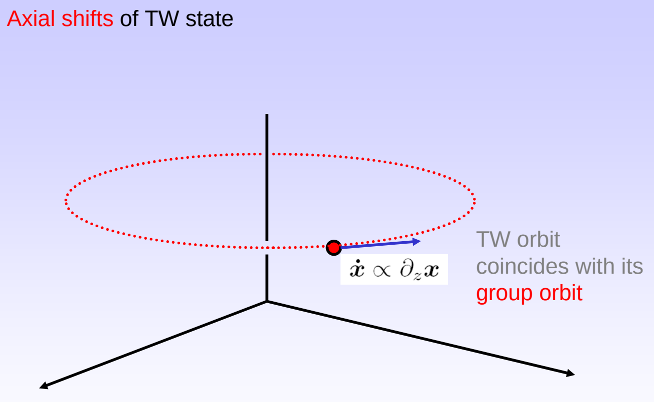
each group orbit an equivalence class

the goal

replace each group orbit by a unique point in a lower-dimensional

symmetry reduced state space $\mathcal{M}/G$

pedestrian attempt : relative equilibrium or 'traveling wave'

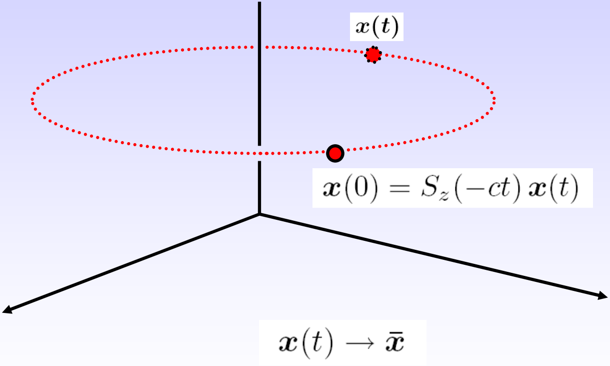


dynamical orbit confined to the group orbit

$$g(\tau)\, x(0) = x(\tau) \in \mathcal{M}_{TW}$$

pedestrian* attempt : moving frame

Reduction of TW orbit to point by shifts



relative equilibrium is made stationary by a counter-rotating 'frame'

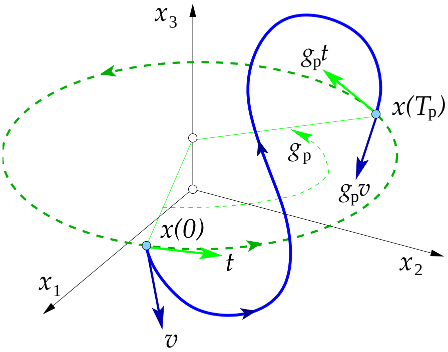
* 'pedestrian' = polite word for 'applied mathematician'

relative periodic orbit A relative periodic orbit p is an orbit in state space $\mathcal{M}$ which exactly recurs

$$x_p(t) = g_p x_p(t + T_p)\,, \qquad x_p(t) \in \mathcal{M}_p$$

for a fixed relative period T_p and a fixed group action $g_p \in G$ that "rotates" the endpoint $x_p(T_p)$ back into the initial point $x_p(0)$.

relative periodic orbit : state space visualization



(green dashes) group orbit
(blue) relative periodic orbit orbit

(arrows) velocity, group tangents

each cycle point

$$x_p(0) = g_p x_p(T_p)$$

exactly recurs at a fixed

relative period

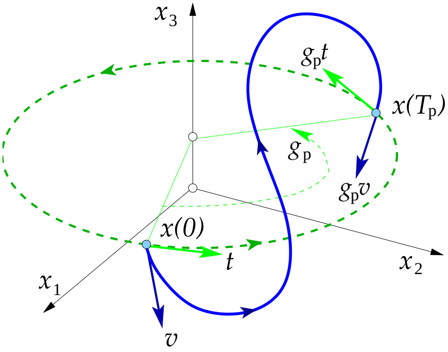
T_p

but shifted by a fixed

group action

g_p

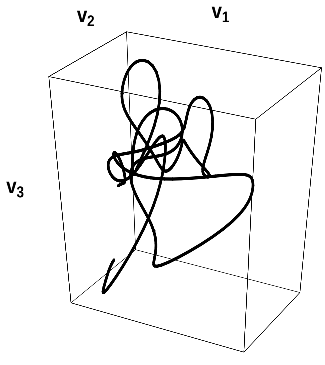
relative periodic orbit : state space visualization



group action parameters $\theta = (\theta_1, \theta_2, \cdots \theta_N)$ are irrational:

trajectory sweeps out ergodically the group orbit without ever closing into a periodic orbit

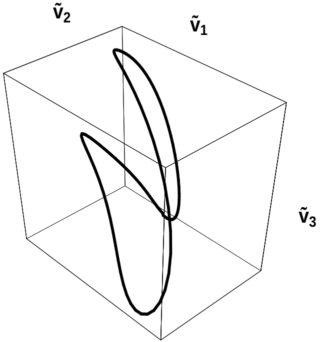
try a co-moving coordinate frame?



a relative periodic orbit of the Kuramoto-Sivashinsky flow, 128*d* state space traced for four periods T_p, projected on

a stationary state space coordinate frame $\{v_1, v_2, v_3\}$; a mess

try a co-moving coordinate frame?

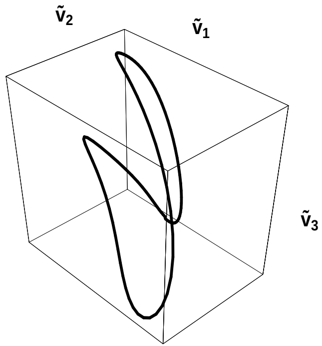


(*b*)

a relative periodic orbit of the Kuramoto-Sivashinsky flow projected on

a co-moving $\{\tilde{v}_1, \tilde{v}_2, \tilde{v}_3\}$ frame

no good global co-moving frame!



(*b*)

beautiful, but this is no symmetry reduction at all;

all other relative periodic orbits require their own frames, moving at different velocities!

relativity for cyclists

method of moving frames / slices

cut group orbits by a hypersurface (kind of Poincaré section), each group orbit of symmetry-equivalent points represented by the single point

cut how?

inspiration: pattern recognition you are observing turbulence in a pipe flow, or your defibrillator has a mesh of sensors measuring electrical currents that cross your heart, and

you have a precomputed pattern, and are sifting through the data set of observed patterns for something like it

here you see a pattern, and there you see a pattern that seems much like the first one

how ‘much like the first one?’

take the first pattern

‘template’ or ‘reference state’

a point $\bar{x}'$ in the state space $\mathcal{M}$

and use the symmetries of the flow to

slide and rotate the ‘template’

act with elements of the symmetry group G on $\bar{x}' \to g(\theta)\,\bar{x}'$

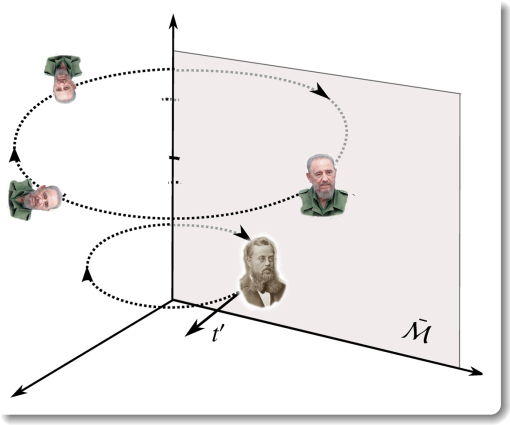
until it overlies the second pattern (a point x in the state space)

distance between the two patterns

$$|x - g(\theta)\,\bar{x}'| = |\bar{x} - \bar{x}'|$$

is minimized

idea: the closest match



template: Sophus Lie

(1) rotate bearded guy x traces out the group orbit $\mathcal{M}_x$

(2) replace the group orbit by the closest match $\bar{x}$ to the template pattern $\bar{x}'$

the closest matches $\bar{x}$ lie in the $(d-N)$ symmetry reduced state space $\bar{\mathcal{M}}$

distance assume that G is a subgroup of the group of orthogonal transformations $\mathrm{O}(d)$, and measure distance $|x|^2 = \langle x|x \rangle$ in terms of the Euclidean inner product

numerical fluids: PDE discretization independent L2 distance is

energy norm

$$\|\mathbf{u}-\mathbf{v}\|^2 = \langle \mathbf{u}-\mathbf{v}|\mathbf{u}-\mathbf{v} \rangle = \frac{1}{V}\int_\Omega d\mathbf{x}\,(\mathbf{u}-\mathbf{v})\cdot(\mathbf{u}-\mathbf{v})$$

experimental fluid:

image discretization independent distance

is Hamming distance, or ???

minimal distance

is a solution to the extremum conditions

$$\frac{\partial}{\partial\theta_a}|x - g(\theta)\,\bar{x}'|^2 = 2\,\langle\bar{x} - \bar{x}'|\bar{t}'_a\rangle = 0\,, \qquad \bar{t}'_a = \mathbf{T}_a\bar{x}'$$

Cartan derivative

$$g^{-1}\dot{g}\,x = e^{-\theta\cdot\mathbf{T}}\,\frac{d}{d\,\tau}e^{\theta\cdot\mathbf{T}}x = \dot{\theta}\cdot t(x)$$

group tangent fields

flow field at the state space point x induced by the action of the group is given by the set of N *tangent fields*

$$t_a(x)_i = (\mathbf{T}_a)_{ij}x_j$$

Lie algebras for pedestrians an element of a compact Lie group:

$$g(\theta) \propto e^{\theta \cdot \mathbf{T}}, \qquad \theta \cdot \mathbf{T} = \sum \theta_a \mathbf{T}_a, \; a = 1, 2, \cdots, N$$

$\theta \cdot \mathbf{T}$: *Lie algebra* element
θ_a: parameters of the transformation.

infinitesimal transformations

$$g = e^{\delta\theta_a T_a} \simeq 1 + \delta\theta_a (T_a), \qquad |\delta\theta| \ll 1$$

Lie algebra

- T_a are generators of infinitesimal transformations
- here T_a are $[d \times d]$ antisymmetric matrices
- T_a are elements of the Lie algebra of G

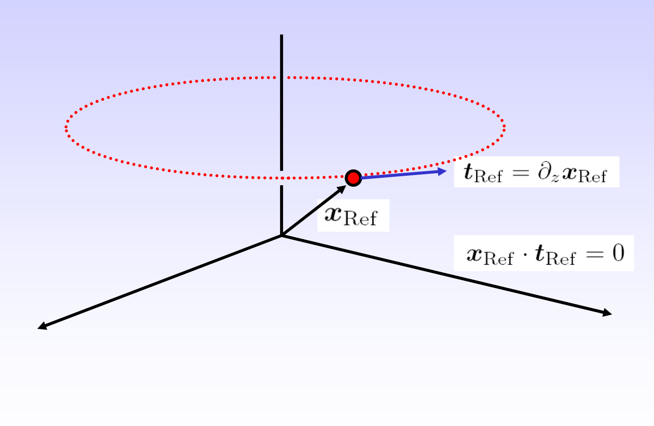
example: $\mathrm{SO}(2)$ invariance of complex Lorenz equations
complex Lorenz equations equations are invariant under $\mathrm{SO}(2)$ rotation by finite angle θ:

$$g(\theta) = \begin{pmatrix} \cos\theta & \sin\theta & 0 & 0 & 0 \\ -\sin\theta & \cos\theta & 0 & 0 & 0 \\ 0 & 0 & \cos\theta & \sin\theta & 0 \\ 0 & 0 & -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

$\mathrm{SO}(2)$ Lie algebra has one generator of infinitesimal rotations

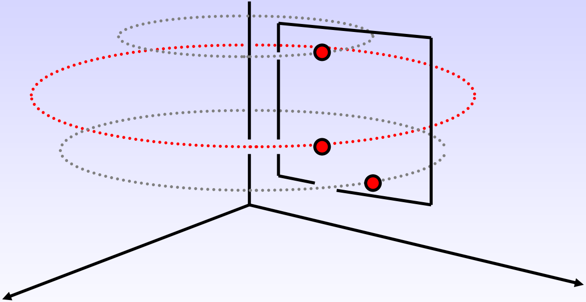
$$\mathbf{T} = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

traveling wave



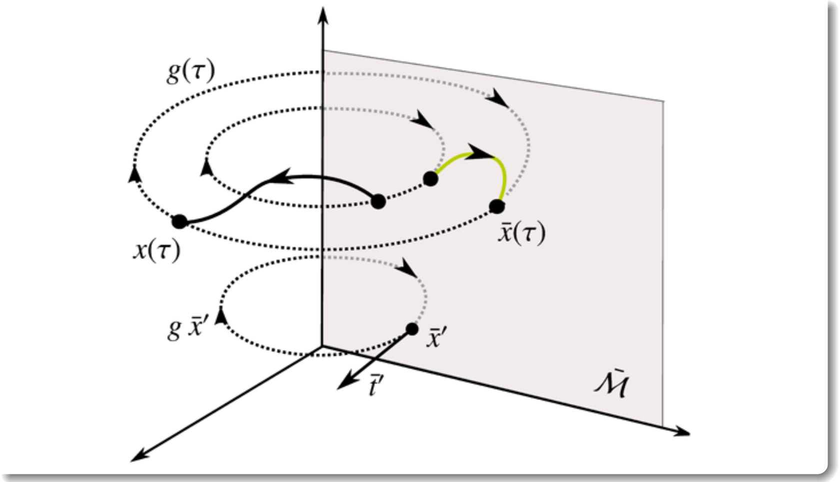
traveling wave

Reduce all TWs into a single slice



$$\boldsymbol{x}_i(0) = S_z(-c_i\, t)\, \boldsymbol{x}_i(t)$$

How? - several speeds *c*, possibly unknown flow within the slice



full-space trajectory $x(\tau)$
rotated into the reduced state space $\bar{x}(\tau) = g(\theta)^{-1}x(\tau)$
by appropriate *moving frame* angles $\{\theta(\tau)\}$

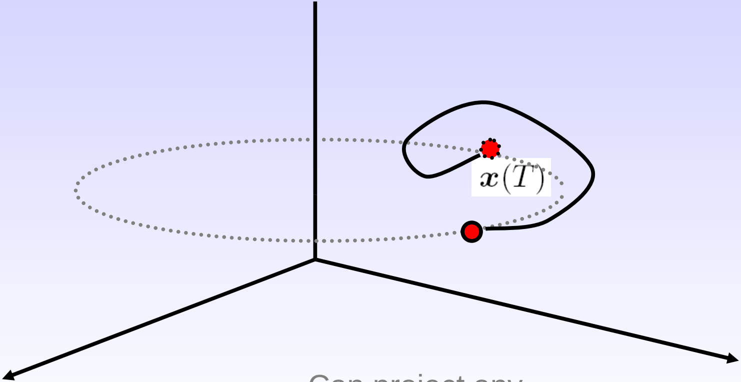
flow within the slice slice fixed by $\bar{x}'$

reduced state space $\bar{\mathcal{M}}$ flow $\bar{v}(\bar{x})$

$$
\begin{aligned}
\bar{v}(\bar{x}) &= v(\bar{x}) - \dot{\theta}(\bar{x}) \cdot t(\bar{x})\,, \qquad \bar{x} \in \bar{\mathcal{M}} \\
\dot{\theta}_a(\bar{x}) &= (v(\bar{x})^T \bar{t}'_a)/(t(\bar{x})^T \cdot \bar{t}')\,.
\end{aligned}
$$

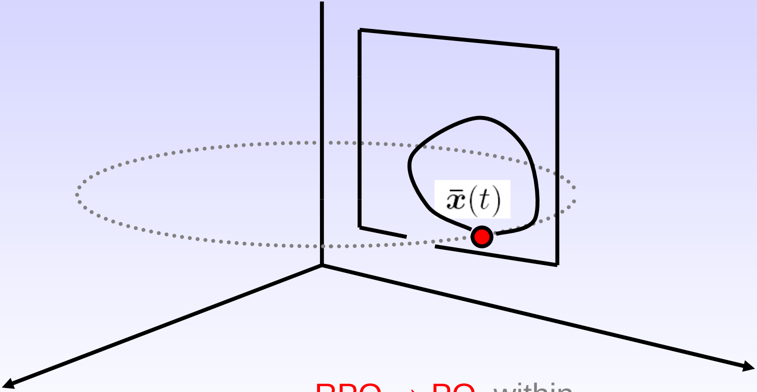
- v : velocity, full space
- $\bar{v}$: velocity component in slice
- $\dot{\theta} \cdot t$: velocity component normal to slice
- $\dot{\theta}$: reconstruction equation for the group phases

Application to a relative periodic orbit (RPO)



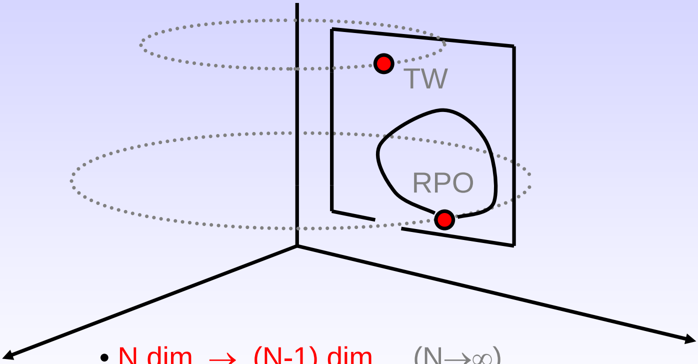
Can project any trajectory into the slice

Application to a relative periodic orbit (RPO)



RPO → PO within the slice

relative equilibria and relative periodic orbits together



- N dim → (N-1) dim (N→∞)
- Automatic removal of strong shift (gives c for TW)
- TW → point
- RPO → PO

symmetry reduction achieved!

- all points equivalent by symmetries are represented by
  - a single point
- families of solutions are mapped to a single solution
  - relative equilibria become equilibria
  - relative periodic orbits become periodic orbits

take-home message rotation into a slice is not an average over 3D pipe azimuthal angle

it is the full snapshot of the flow embedded in the

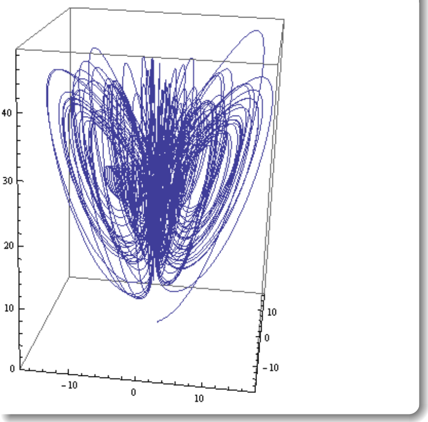
∞-dimensional state space

NO information is lost by symmetry reduction

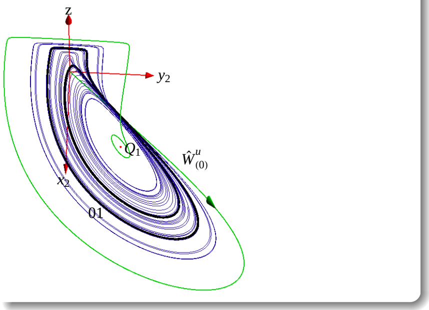
- not modeling by a few degrees of freedom
- no dimensional reduction

die Lösung : complex Lorenz flow reduced

full state space

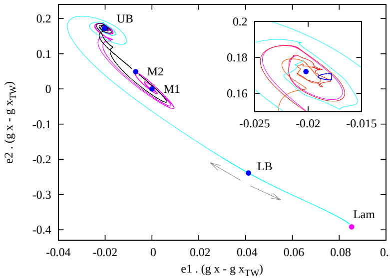


reduced state space

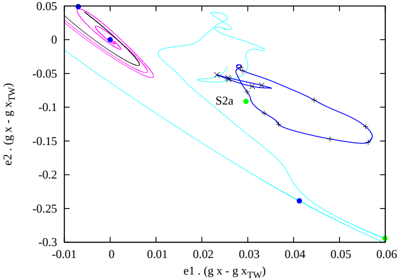


ergodic trajectory was a mess, now the topology is reveled
relative periodic orbit $\overline{01}$ now a periodic orbit

triumph : all pipe flow solution in one happy family $\bar{x}'$ is typical turbulent state (breaks all symmetries)
plot $N2\_M1, \cdots$, relative equilibria, unstable manifolds



all in the same projection
inset: an expanded view
blue loop: $T = 4.93$
relative periodic orbit

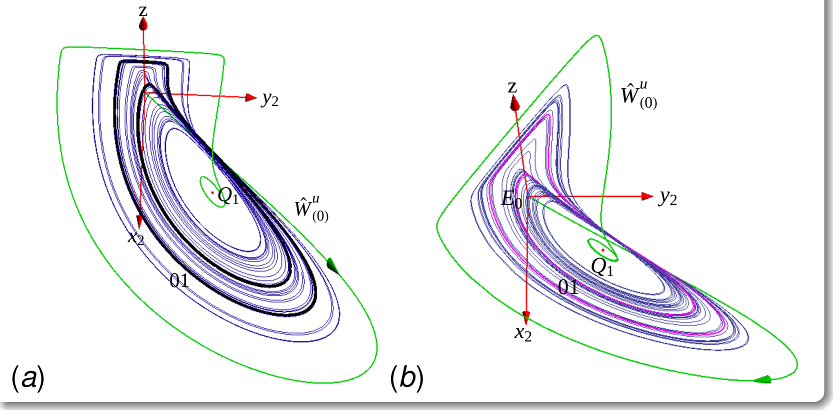


$T = 10.96$ and $T = 36.92$
relative periodic orbits
embedded in turbulence

first 'turbulent' relative periodic orbits for pipe flows!

slice trouble 1

portrait of complex Lorenz flow in reduced state space



any choices of the slice $\bar{x}'$ exhibit flow discontinuities

slice trouble 1

glitches!

group tangent of a generic trajectory orthogonal to the slice tangent at a sequence of instants τ_k

$$t(\tau_k)^T \cdot \bar{t}' = 0$$

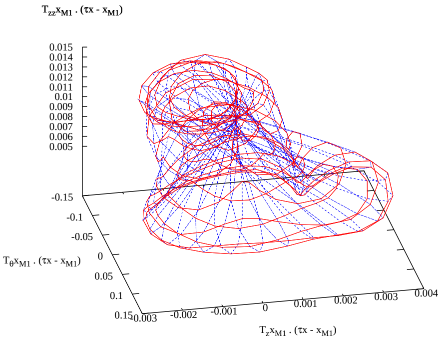
example: group orbit of a pipe flow turbulent state $\bar{x}'$ is Kerswell *et al* $N2\_M1$ relative equilibrium
($Re = 2400$, stubby $L = 2.5D$ pipe)

$\mathrm{SO}(2) \times \mathrm{SO}(2)$ symmetry
$\Rightarrow$ group orbit is 2-torus

a turbulent state

distance extremum condition

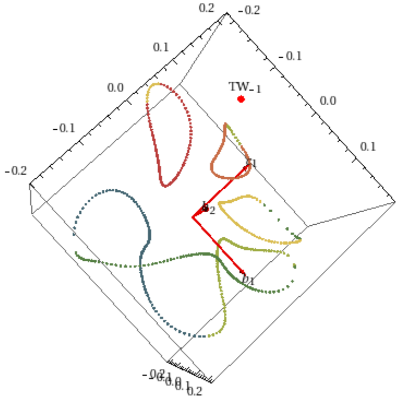
$$\frac{\partial}{\partial \theta_a} |x - g(\theta)\, \bar{x}'|^2 = 0$$



group orbits of highly nonlinear states are highly contorted: many extrema, multiple sections by a slice

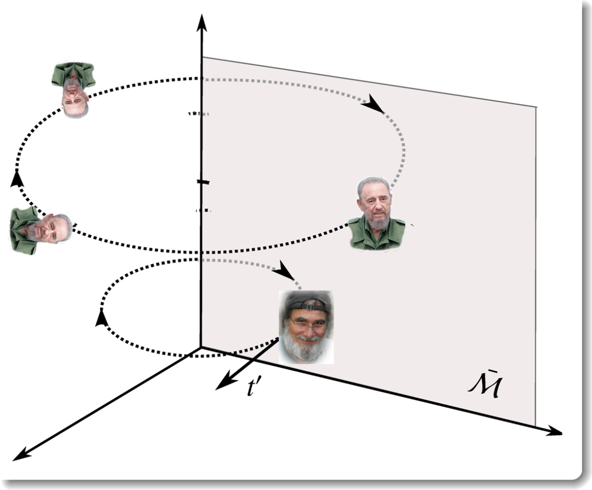
slice trouble 2

slice cuts a relative periodic orbit multiple times



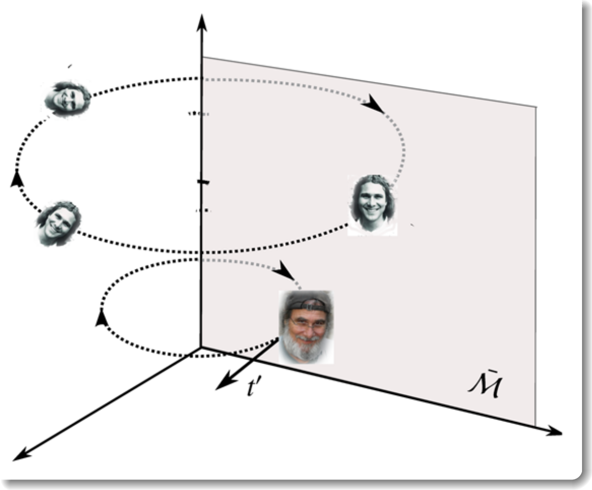
a single relative periodic orbit is intersected by a slice in 3 separate sections of the relative periodic orbit torus and 3 sections that appear to connect to a closed loop

trouble: slices cannot be global



representing a group orbit by the closest match to a good template $\bar{x}'$ (Phil Morrison)

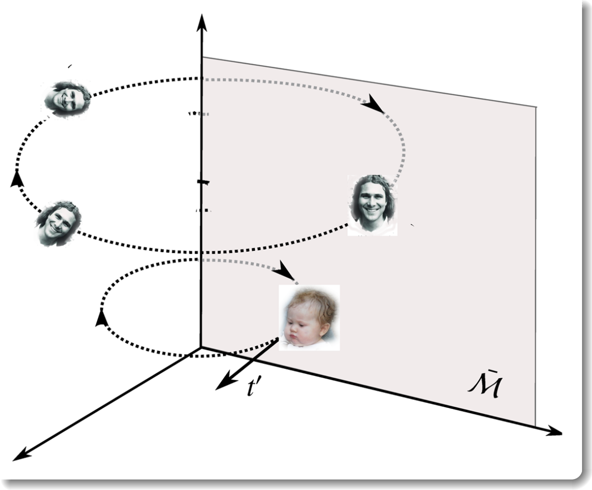
trouble: slices cannot be global



the 'closest match' to a bad template $\bar{x}'$ (young Phil Morrison) can be a mismatch

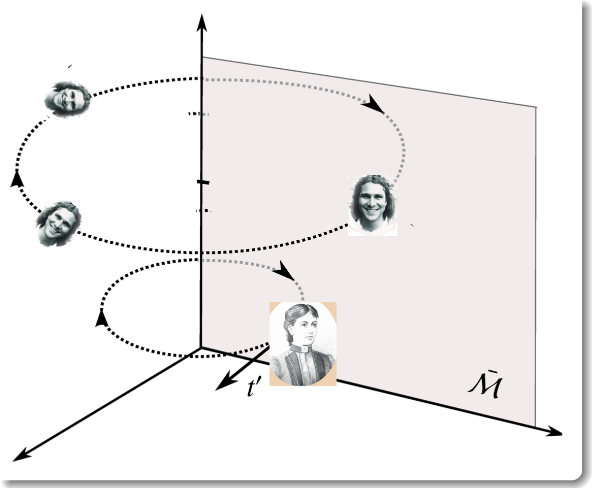
single template cannot be a good match globally

trouble: slices cannot be global



another attempt,
another bad
template $\bar{x}'$
(younger Morrison)

trouble: slices cannot be global



representing a group orbit by the closest match to a better template $\bar{x}'$ (Sonya Kovalewskaya)

to cover $\mathcal{M}/G$ globally, need:
a set of templates:

- 2 rolls
- 4 rolls
- ...

How good is your slice? hyperplane of points x^* defined by being normal to the quadratic Casimir-weighted vector $\mathbf{T}^2 \bar{x}'$, such that from the template vantage point their group orbits are not transverse, but locally 'horizontal,'

$$\langle t(x^*) | \bar{t}' \rangle = -\langle x^* | \mathbf{T}^2 \bar{x}' \rangle = 0$$

(for simplicity, specialize to the $\mathrm{SO}(2)$ case) inflection hyperplane S : set of all points $\bar{x}^*$ which are both

(a) in the slice

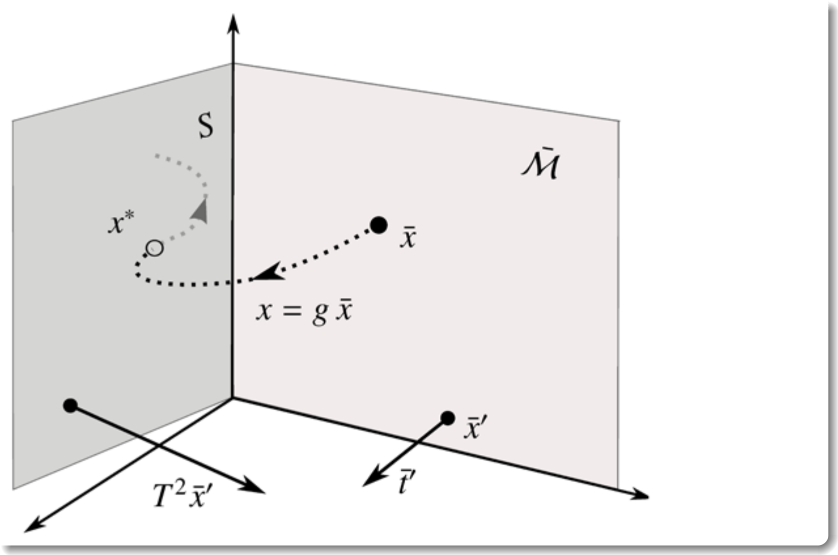
(b) whose group tangent $t(\bar{x}^*)$ is also in the slice

$$\begin{aligned} \langle \bar{x}^* | \bar{t}' \rangle &= 0 \\ \langle t(\bar{x}^*) | \bar{t}' \rangle &= -\langle \bar{x}^* | \mathbf{T}^2 \bar{x}' \rangle = 0 \end{aligned}$$

S is the locus of inflection points, a hyperplane through which

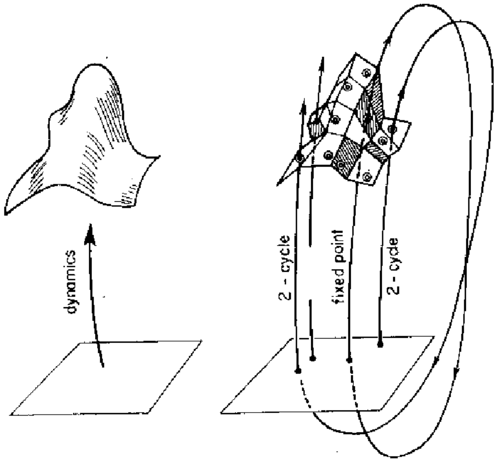
- curvature of the distance function changes sign
- local minimum turns into a local maximum

slice is good up to inflection hyperplane



charting the state space for turbulent/chaotic systems a set of Poincaré sections is needed to capture the dynamics. The choice of sections should reflect the dynamically dominant patterns seen in the solutions of nonlinear PDEs

we propose to construct a global atlas of the dimensionally reduced state space $\bar{\mathcal{M}}$ by deploying linear Poincaré sections $\mathcal{P}^{(j)}$ across neighborhoods of the qualitatively most important patterns $\bar{x}'^{(j)}$



this is the periodic-orbit implementation of the idea of state space tessellation we shall refer to these states as *templates*, each represented in the state space $\mathcal{M}$ of the system by a *template point* $\bar{x}'$

together with the velocity field at this point, a template defines a linear Poincaré section, an affine hyperplane $\bar{x} \in \mathcal{P}$,

$$v(\bar{x}') \cdot (\bar{x} - \bar{x}') = 0\,,$$

locally normal to the $v(\bar{x}')$ at the template point $\bar{x}'$

each Poincaré section $\mathcal{P}^{(j)}$, provides a local chart at $\bar{x}'^{(j)}$ for a neighborhood of an important, qualitatively distinct class of solutions (2-rolls states, 3-rolls states, etc.)

together they ‘Voronoi’ tessellate the curved manifold in which the reduced strange attractor is embedded by a finite set of hyperplane tiles a Poincaré section is a $(d-1)$-dimensional hyperplane. If we pick another template point $\bar{x}'^{(2)}$, it comes along with its own Poincaré section

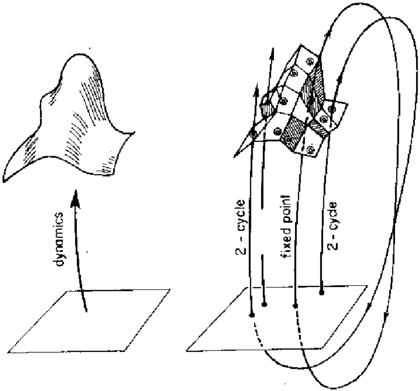
any neighboring pair of $(d-1)$-dimensional Poincaré sections intersects in a ‘ridge’ (‘boundary,’ ‘edge’), a $(d-2)$-dimensional hyperplane, easy to compute a global atlas so constructed should be sufficiently fine-grained: each ‘chart’ or ‘tile,’ bounded by ridges to neighboring Poincaré sections, should be sufficiently small

physical task is to, for a given dynamical flow, pick a set of qualitatively distinct templates whose Poincaré sections are locally tangent to the strange attractor

state space tessellation by periodic orbits

two 1-cycles

a 2-cycle that alternates between the neighborhoods of the two 1-cycles, shadowing first one of the two 1-cycles, and then the other



smooth dynamics (left frame) tesselated by the skeleton of periodic points, together with their linearized neighborhoods, (right frame)

summary

conclusion

- symmetry reduction by method of slices:
  efficient, allows exploration of high-dimensional flows hitherto unthinkable
- stretching and folding of unstable manifolds in reduced state space organizes the flow

to be done

- construct Poincaré sections and return maps
- find all (relative) periodic orbits up to a given period
- use the information quantitatively (periodic orbit theory)

take-home message if you have a symmetry

use it!

without symmetry reduction, no understanding of pipe, Couette, ..., flows possible amazing theory! amazing numerics! frustration...

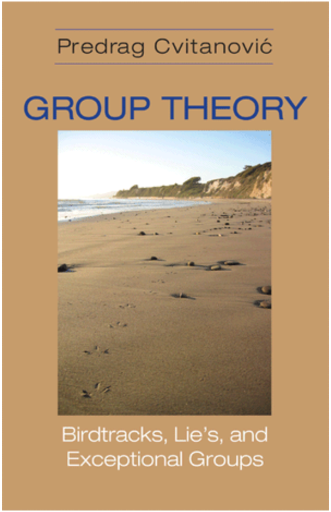
"Ask your doctor if taking a pill to solve all your problems is right for you."

Happy families are all alike;
every unhappy family is unhappy in its own way

everybody, her mother,
and Robert MacKay knows how to do this

except the author of

masters of group theory



Lie groups all the group theory needed here is in principle contained in the Peter-Weyl theorem:

compact Lie group *G*

- completely reducible
- its representations are fully reducible
- every compact Lie group is a subgroup of $U(n)$ for some n
- every continuous, unitary, irreducible representation of a compact Lie group is finite dimensional

example: discrete symmetries of plane Couette flow
Navier-Stokes equations for the plane Couette flow have two discrete symmetries, in addition to the two continuous translational symmetries: rotation by π in the (streamwise,spanwise) plane, and rotation by π in the (streamwise,wall-normal) plane

that is why there exist equilibria (as opposed to relative equilibria) and some periodic orbits in the plane Couette flow

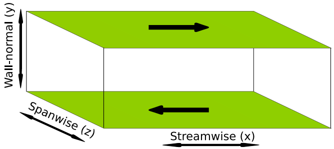
they belong to discrete symmetry subspaces

a relative periodic orbit can be pre-periodic if it is invariant under a discrete symmetry: If $g^m = 1$ is of finite order m, then the corresponding orbit is periodic with period mT_p.

example: group orbits of plane Couette flow

plane Couette flow

flow periodic in streamwise and spanwise directions



- $SO(2) \times SO(2)$ continuous symmetry
- each equilibrium/relative equilibrium $\in$ 2-torus of equivalent solutions
- a relative periodic solution recurs at time T_p with exactly the same disposition of velocity fields over the entire box, but shifted by a 2-dimensional (streamwise,spanwise) translation g_p

group action g_p is referred to as a “phase,” or a “shift”

continuous symmetry parameters $\{t, \theta_1, \cdots, \theta_N\}$ are real numbers, ratios π/θ_j are almost never rational, and relative periodic orbits are almost never eventually periodic

$\bar{x}$ is the point on the group orbit of x

$$x = g(\theta)\,\bar{x}\,, \qquad g \in G\,,$$

closest to the template $\bar{x}'$, the Lie group element $g = g(\theta) \propto \exp\left(\theta \cdot \mathbf{T}\right)$

symmetry reduction methods the key result of the representation theory of invariant functions

Hilbert-Weyl theorem

For a compact group G there exist a finite G-invariant homogenous polynomial basis $\{u_1, u_2, \ldots, u_m\}$ such that any G-invariant polynomial can be written as a multinomial

$$h(x) = p(u_1(x), u_2(x), \ldots, u_m(x)) \,.$$

explicit construction of such basis is often not feasible, and we will not take this path except for a few simple low-dimensional cases we propose to construct a global atlas by deploying a set of linear Poincaré sections and slices in neighborhoods of the most important equilibria and/or periodic orbits as local charts reduction methods

1. **Hilbert polynomial basis**: rewrite equivariant dynamics in invariant coordinates
2. **moving frames, or slices**: cut group orbits by a hypersurface (kind of Poincaré section), each group orbit of symmetry-equivalent points represented by the single point

reduction methods

1. **Hilbert polynomial basis**: rewrite equivariant dynamics in invariant coordinates: global, but useless in dimension larger than 10
2. **moving frames, or slices**: cut group orbits by a hypersurface (kind of Poincaré section), each group orbit of symmetry-equivalent points represented by the single point: local

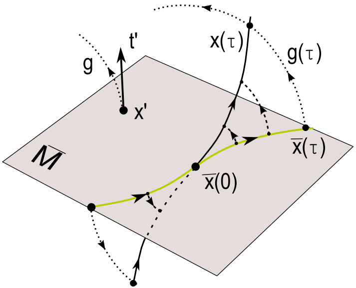
moving frame angles θ_i computed $g(\theta)\,\bar{x} = x$ 'angles' are fixed by

$$\langle \bar{x} | \bar{t}'_a \rangle = 0$$

the set of *extremal* group orbit points lies in a $(d-N)$-dimensional hyperplane, the set of vectors orthogonal to the template tangent space spanned by tangent vectors $\{\bar{t}'_1, \cdots, \bar{t}'_N\}$

slice & dice

flow reduced to a slice



slice $\bar{\mathcal{M}}$ through the slice-fixing point $\bar{x}'$, normal to the group tangent $\bar{t}'$ at $\bar{x}'$, intersects group orbits (dotted lines). The full state space trajectory $x(\tau)$ and the reduced state space trajectory $\bar{x}(\tau)$ are equivalent up to a group rotation $g(\tau)$.