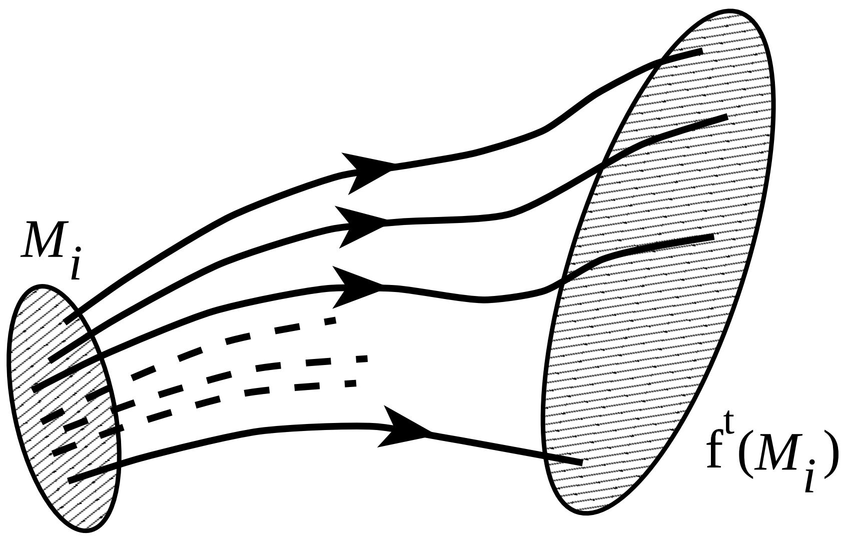
Chapter 4
Local stability

Flows transport neighborhoods



next: transport the neighborhood of x(t)

Equations of variations

Flow transports displacement $x(t) + \delta x(t)$
along trajectory $x(t) = f^t(x_0)$.

equations of variations for infinitesimal neighborahood:

$$\dot{x}_i + \dot{\delta x}_i = v_i(x + \delta x) \approx v_i(x) + \sum_j \frac{\partial v_i}{\partial x_j}\delta x_j \,.$$

Together

$$\dot{x}_i = v_i(x)\,, \qquad \dot{\delta x}_i = \sum_j A_{ij}(x)\delta x_j$$

where stability matrix

$$A_{ij}(x) = \frac{\partial v_i(x)}{\partial x_j}$$

is the instantaneous rate of shearing of $x(t)$ neighborhood

fundamental matrix

infinitesimal neighborhood after a finite time:

$$f_i^t(x_0 + \delta x) = f_i^t(x_0) + \sum_j \frac{\partial f_i^t(x_0)}{\partial x_{0j}} \delta x_j + \cdots ,$$

Linearized neighborhood transported by the Jacobian

$$\delta x(t) = J^t(x_0)\delta x(0) , \qquad J_{ij}^t(x_0) = \left. \frac{\partial x_i(t)}{\partial x_j} \right|_{x=x_0} .$$

Stability of trajectories

exponential of a constant matrix

$$e^{t\mathbf{A}} = \lim_{m\to\infty}\left(\mathbf{1} + \frac{t}{m}\mathbf{A}\right)^m .$$

tax-accountant's discrete step definition of an exponential

local rate of neighborhood distortion $\mathbf{A}(x)$ depends on $x(t)$

$$\begin{aligned} J^t &= \lim_{m\to\infty}\prod_{n=m}^{1}\left(\mathbf{1} + \delta t\,\mathbf{A}(x_n)\right) \\ &= \lim_{m\to\infty} e^{\delta t\,\mathbf{A}(x_n)} e^{\delta t\,\mathbf{A}(x_{m-1})}\cdots e^{\delta t\,\mathbf{A}(x_2)} e^{\delta t\,\mathbf{A}(x_1)} , \\ &\qquad \delta t = (t - t_0)/m, x_n = x(t_0 + n\delta t). \end{aligned}$$

limit of this procedure:

$$J^t_{ij}(x_0) = \left[\mathbf{T} e^{\int_0^t d\tau \mathbf{A}(x(\tau))} \right]_{ij} ,$$

where $\mathbf{T}$ stands for time-ordered integration.

the fundamental matrices are multiplicative along the flow,

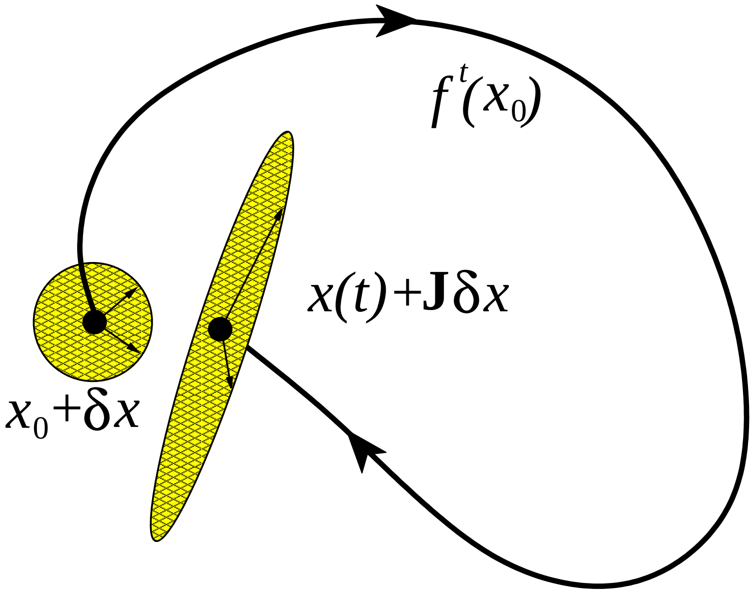
$$J^{t+t'}(x) = J^{t'}(x')J^t(x), \qquad \text{where } x' = f^t(x) .$$

Λ_k = kth stability eigenvalue of the finite time fundamental matrix M^t.

λ_k = kth stability exponent

$$|\Lambda_k| = e^{t\lambda_k} , \qquad \Lambda_k = \Lambda_k(x_0, t) , \; \lambda_k = \lambda_k(x_0, t) .$$

fundamental matrix



Jacobian maps a spherical neighborhood of x_0 into an ellipsoidal neighborhood time t later

Neighbors separate along unstable directions, approach each other along stable directions, creep along the marginal directions

Stability of equilibria

Stability matrix $\mathbf{A} = \mathbf{A}(x_q)$ evaluated at an equilibrium point x_q is constant

$$f^t(x) = x_q + e^{\mathbf{A}t}(x - x_q) + \cdots ,$$
$$J^t(x_q) = e^{\mathbf{A}t} \qquad \mathbf{A} = \mathbf{A}(x_q) .$$

For a constant $\mathbf{A}$ the fundamental matrix

$$x(t) = e^{t\mathbf{A}}x(0) .$$

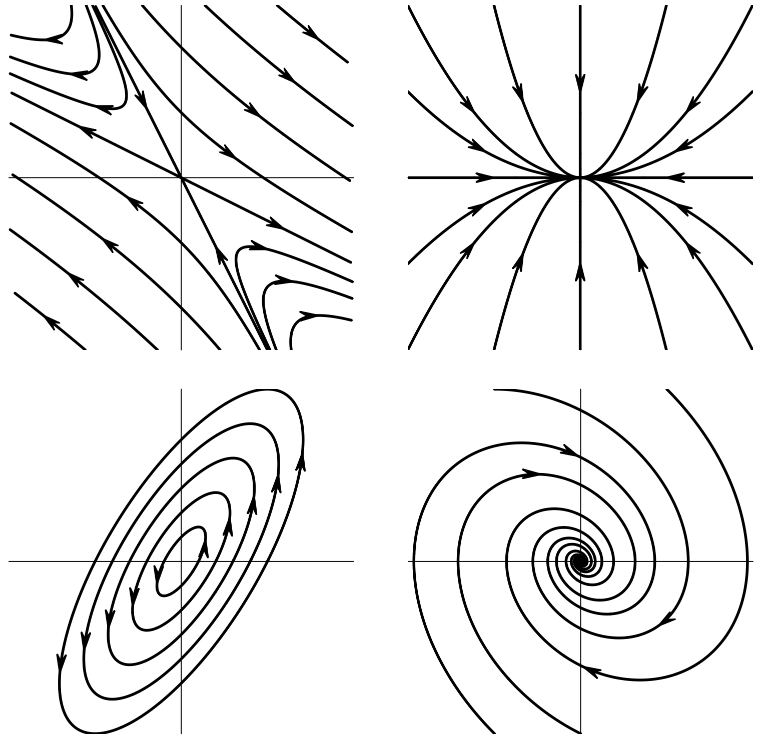
is the solution of the linear equation

$$\dot{x} = \mathbf{A}x .$$

so study linear flows first:

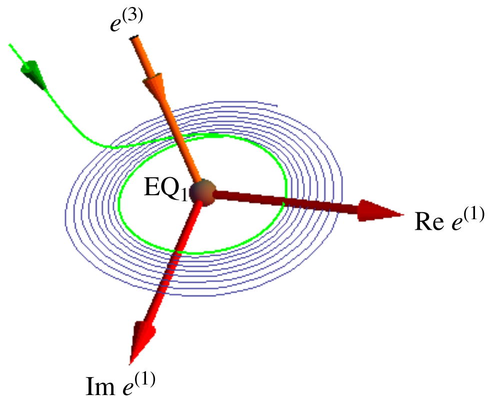
2-dimensional flows

Streamlines for several typical 2-dimensional flows:
saddle (hyperbolic),
in node (attracting),
center (elliptic),
in spiral.

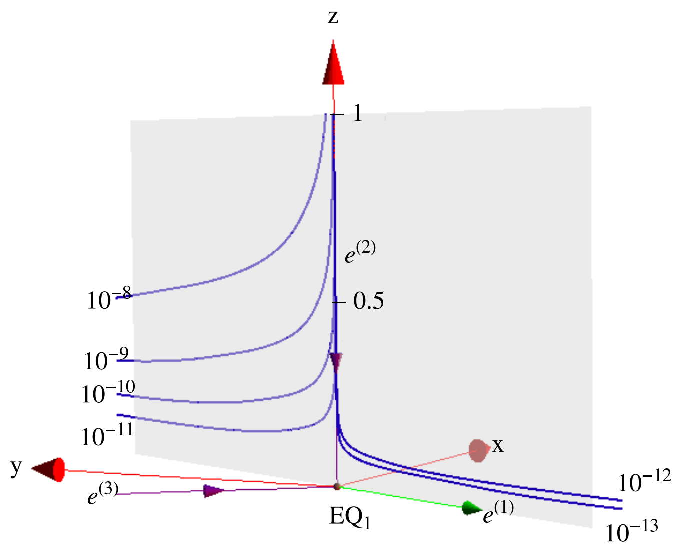


Stability of Lorenz flow equilibria

the flow is organized by its 3 equilibria



(a)



(b)

(a) The unstable eigenplane spanned by Re $\mathbf{e}^{(1)}$ and Im $\mathbf{e}^{(1)}$, the stable eigenvector $\mathbf{e}^{(3)}$, EQ$_1$ equilibrium.

(b) flow near the EQ$_0$ equilibrium: unstable eigenvector $\mathbf{e}^{(1)}$, stable eigenvectors $\mathbf{e}^{(2)}$, $\mathbf{e}^{(3)}$. Note the strong $\lambda^{(1)}$ expansion: the

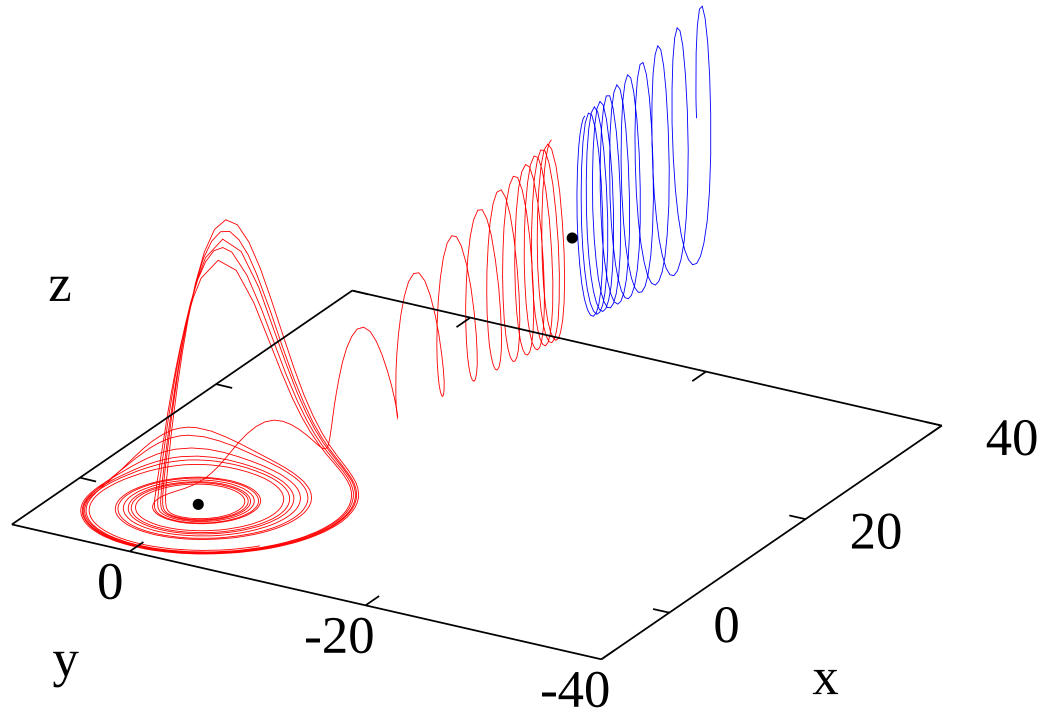
EQ_0 equilibrium is unreachable, and the $EQ_1 \to EQ_0$ heteroclinic connection never observed in simulations.

(E. Siminos)

Stability of Rössler flow equilibria

two equilibrium points (x^-, y^-, z^-) (x^+, y^+, z^+)

stable manifold of "+" equilibrium point = attraction basin boundary:



right of the "+" equilibrium trajectories escape,

left of the "+" spiral toward the "−" equilibrium point

→ seem to wander chaotically for all times.

linearized stability exponents

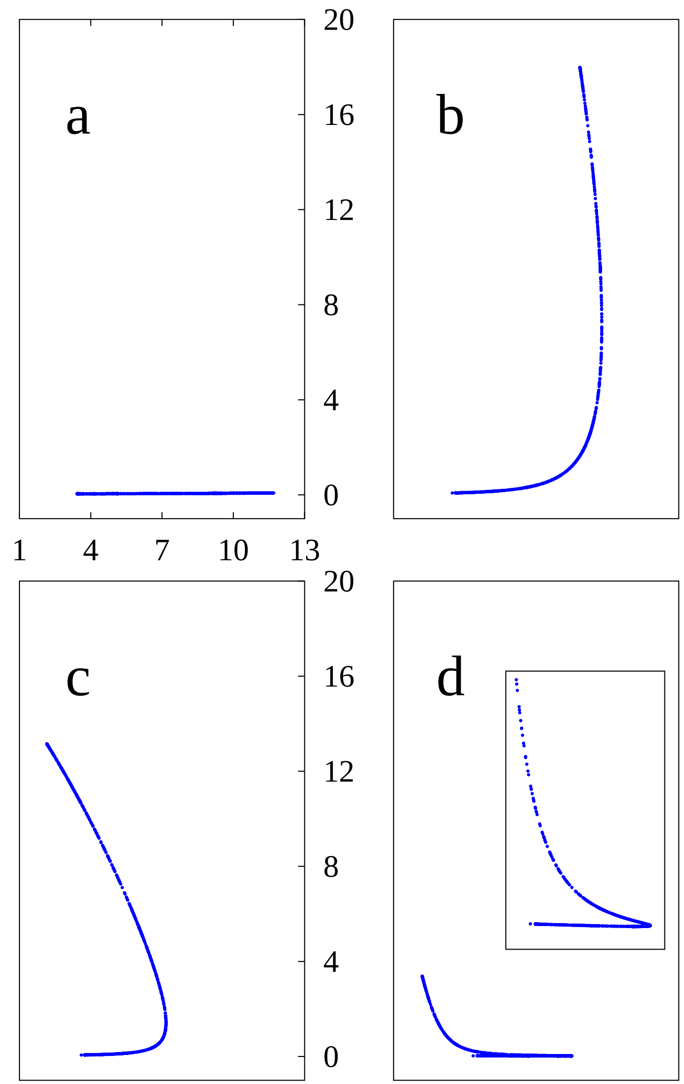
$$(\lambda_1^-, \lambda_2^- \pm i\vartheta_2^-) = (-5.686, \quad 0.0970 \pm i\,0.9951)$$
$$(\lambda_1^+, \lambda_2^+ \pm i\vartheta_2^+) = (0.1929, \quad -4.596 \times 10^{-6} \pm i\,5.428)$$

The $\lambda_2^- \pm i\vartheta_2^-$ eigenvectors span a plane

this plane rotates with angular period $T_- \approx \left|2\pi/\vartheta_2^-\right| = 6.313$

a trajectory that starts near the "−" equilibrium point spirals away per one rotation with multiplier $\Lambda_{radial} \approx \exp(\lambda_2^- T_-) = 1.84$

each Poincaré section return, contracted into the stable manifold by amazing factor of $\Lambda_1 \approx \exp(\lambda_1^- T_-) = 10^{-15.6}$ (!)



start with a 1 mm interval pointing in the contracting Λ_1 eigendirection.

After one Poincaré return the interval is of order of 10^{-4} fermi

Rössler Poincaré return map is in practice 1 − dimensional