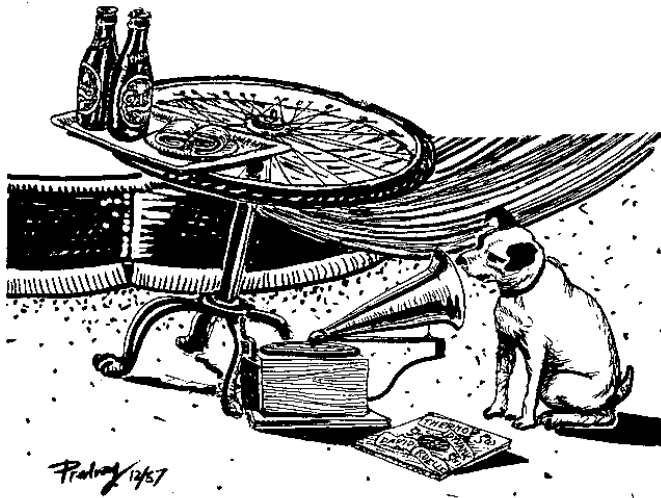


# Chaos: Classical and Quantum

## I: Deterministic Chaos



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## Contributors

No man but a blockhead ever wrote except for money  
—Samuel Johnson

This book is a result of collaborative labors of many people over a span of several decades. Coauthors of a chapter or a section are indicated in the byline to the chapter/section title. If you are referring to a specific coauthored section rather than the entire book, cite it as (for example):

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Daniel Borrero Oct 23 2008, [soluCycles.tex](#)

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I feel I never want to write another book. What's the good!  
I can eke living on stories and little articles, that don't cost  
a tithe of the output a book costs. Why write novels any  
more!

—D.H. Lawrence

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Who is the 3-legged dog reappearing throughout the book? Long ago, when we were innocent and knew not Borel measurable  $\alpha$  to  $\Omega$  sets, P. Cvitanović asked V. Baladi a question about dynamical zeta functions, who then asked J.-P. Eckmann, who then asked D. Ruelle. The answer was transmitted back: "The

master says: 'It is holomorphic in a strip.'" Hence His Master's Voice logo, and the 3-legged dog is us, still eager to fetch the bone. The answer has made it to the book, though not precisely in His Master's voice. As a matter of fact, the answer *is* the book. We are still chewing on it.

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