3. PATH INTEGRALS

An inconvenient aspect of the generating functional formalism is the proliferation of derivatives. Green function legs are pulled out by taking derivatives with respect to sources, equation (2.11), so that the Dyson-Schwinger equations are differential equations. This is a familiar problem. It is usually resolved by finding a transformation (such as Fourier transform) which diagonalizes the differential operators (for example, maps $\frac{d}{dx^\mu} \to k_\mu$). For generating functionals such transformation is called a <u>path integral</u>.

Path integrals have many virtues: they make the symmetries of the theory explicit, they help identify physically dominant configurations, and they suggest systematic ways of computing the quantum corrections to the classically dominant configurations (the saddlepoint expansion). Sometimes they can even be evaluated directly, without resorting to perturbative expansions, by Monte Carlo methods.

A. A Fourier transform

To illustrate the idea, let us get rid of $\frac{d}{dJ_i}$ derivatives by going from generating functionals to their Fourier transforms:

$$Z[J] = \int [d\phi] \tilde{Z}[\phi] e^{i\phi_i J_i} \quad , \tag{3.1}$$

$$[d\phi] = \frac{d\phi_1}{\sqrt{2\pi}} \frac{d\phi_2}{\sqrt{2\pi}} \cdots \quad , \tag{3.2}$$

$$- i\frac{d}{dJ_i} Z[J] = \int [d\phi] \phi_i \tilde{Z}[\phi] e^{i\phi_i J_i} \; . \tag{3.3}$$

Fields ϕ_i are dual to sources J_i in the same sense that momenta k^μ are dual to space coordinates x^μ. As the indices i, j, ... can take continuous values, these integrals are functional integrals. $\tilde{Z}[\phi]$ can be determined by taking a Fourier transform of the DS equation (2.15):

$$0 = \int [d\phi] \left(\frac{dS[\phi]}{d\phi_i} + J_i \right) \tilde{Z}[\phi] e^{i\phi_i J_i} \quad ,$$

$$-i \frac{d}{d\phi_i} \tilde{Z}[\phi] = \frac{dS[\phi]}{d\phi_i} \tilde{Z}[\phi] \ .$$

This is again an easy differential equation to solve. The solution is called the path integral representation of generating functionals:

$$Z[J] = \int [d\phi] e^{i(S[\phi]+\phi_i J_i)} \ . \tag{3.4}$$

In this "derivation" we were rather cavalier about factors of "i" and questions of convergence. As Jens, the serious young student of field theory, objects, we try one more time.

B. Gaussian integrals

It has probably not escaped your notice that the only integral an average physicist can do is the Gaussian integral

$$\int [d\phi] e^{-\frac{\phi^2}{2\lambda}} = \sqrt{\lambda} \ , \qquad [d\phi] = \frac{d\phi}{\sqrt{2\pi}} \ . \tag{3.5}$$

This is the Gaussian integral in one dimension. In more dimensions, Gaussian integrals make their appearance in a slightly jazzed-up form

$$\int [d\phi] e^{-\frac{1}{2}\phi_i \Delta^{-1}_{ij} \phi_j} = \sqrt{\mathrm{Det}\,\Delta} \quad . \tag{3.6}$$

Derivation: Take $\Delta_{ij} = \Delta_{ji}$. A real symmetric matrix can be diagonalized by a rotation R:

$$(R^{-1}\Delta R)_{ij} = \lambda_i \delta_{ij} \ .$$

Volume is rotationally invariant: $[d(R\phi)]=[d\phi]$. Diagonalization reduces the integral to a product of one-dimensional integrals (3.5):

$$\prod_i \int \frac{d\phi_i}{\sqrt{2\pi}} e^{-\frac{\phi_i^2}{2\lambda_i}} = \prod_i \lambda_i^{\frac{1}{2}}$$

The result can be expressed as a determinant:

$$\mathrm{Det}\,\Delta = \mathrm{Det}(R^{-1}\Delta R) = \begin{Vmatrix} \lambda_1 & & & 0 \\ & \lambda_2 & & \\ & & \lambda_3 & \\ 0 & & & \ddots \end{Vmatrix} = \prod_i \lambda_i \quad .$$

Using the invariance of the measure under translation $\phi_i \to \phi_i + J_k\Delta_{ki}$, we can add sources and rederive the generating functional (2.23) for the free field theory:

$$Z_0[J] = \int [d\phi] e^{-\frac{1}{2}\phi_i \Delta_{ij}^{-1}\phi_j + \phi_i J_i} = \sqrt{\mathrm{Det}\,\Delta}\, e^{\frac{1}{2}J_i \Delta_{ij} J_j} \quad . \tag{3.7}$$

The square-root factor is an overall normalization (vacuum bubbles) which does not contribute to the connected diagrams and is (in this case) without physical significance. Remember that the collective index <u>i</u> can take both discrete and continuous values; (3.6) is the definition of the <u>functional</u> Gaussian integral.

The point of this whole exercise is that Gaussian integrals give us the desired fields-sources duality:

$$\frac{d}{dJ_i} \int [d\phi] e^{-\frac{\varepsilon\phi^2}{2} + \phi_i J_i} = \int [d\phi] \phi_i e^{-\frac{\varepsilon\phi^2}{2} + \phi_i J_i} \quad . \tag{3.8}$$

Now we can go back to our definition of the path integral, and make it slightly more respectable by introducing a Gaussian damping factor:

$$Z[J] = e^{S[\frac{d}{dJ}]} \int [d\phi] e^{-\frac{\varepsilon\phi^2}{2} + \phi_i J_i} \quad , \qquad \varepsilon \to 0_+ \quad .$$

This defines the path integral, at least as a formal power series in ϕ or d/dJ:

$$Z[J] = \int [d\phi] e^{S[\phi] + \phi_i J_i} \quad , \tag{3.9}$$

irrespective of whether the action is real or imaginary, or whether we have statistical or quantum mechanics in mind. In the above we have absorbed the damping factor into propagators:

$$S[\phi] = -\tfrac{1}{2}\,\phi_i (\Delta_{ij}^{-1} + \varepsilon)\phi_j + S_I[\phi] \quad . \tag{3.10}$$

This gives the correct imaginary parts for Feynman propagators in Minkowski space (this prescription is sometimes referred to as the Euclidicity postulate).

It will become quite apparent in the discussion of fermionic Green functions that the path integrals should not be taken too literally as "integrals". They are mostly tricks for replacing differential operators $\frac{d}{dJ}$ by number-valued fields ϕ. That should not give you sleepless nights. The history of the subject is that the problems are almost always first recognized and solved in the diagrammatic formalism, and later formulated elegantly in the language of path integrals.

In the path integral formalism, the full Green functions are field expectation values:

$$G_{ij..k} = \langle\phi_i\phi_j..\phi_k\rangle = \int[d\phi]\phi_i\phi_j..\phi_k \, e^{S[\phi]} \Big/ \int[d\phi]e^{S[\phi]} \quad . \tag{3.11}$$

In statistical mechanics, the action is a real number which assigns the probability (the Boltzmann weight) to a given field configuration. In quantum mechanics, the action is an imaginary phase which determines the amplitude of a given field configuration.

<u>Exercise 3.B.1</u> Extend Gaussian integration to complex fields

$$\psi_k = \frac{1}{\sqrt{2}}\left(\phi_{2k-1} + i\phi_{2k}\right), \; \psi^k = \frac{1}{\sqrt{2}}\left(\phi_{2k-1} - i\phi_{2k}\right) .$$

Take the propagator Δ^i_j to be a hermitian matrix. Show that for complex fields the free field generating functional (3.7) is given by

$$\begin{aligned} Z[\eta,\bar{\eta}] &= \int[d\psi d\bar{\psi}]e^{-\bar{\psi}\Delta^{-1}\psi + \bar{\eta}\psi + \bar{\psi}\eta} \\ &= \mathrm{Det}\,\Delta \, e^{\bar{\eta}\Delta\eta} \quad , \end{aligned} \tag{3.12}$$

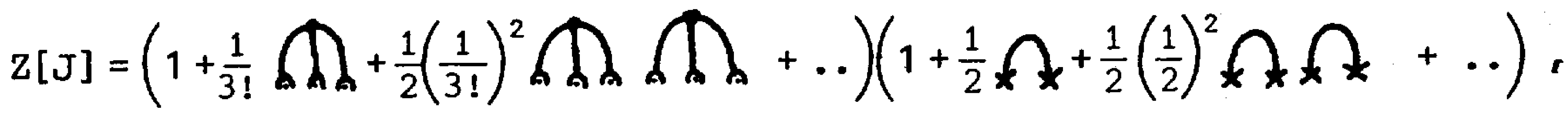
where η_k, η^k are complex sources.

C. Wick expansion

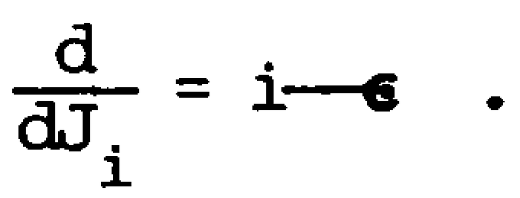
Splitting of the action into a quadratic part and an interaction part, as in (2.13) and (3.10), provides another way of generating the perturbation expansion:

$$Z[J] = \int[d\phi]e^{S_I[\phi] - \frac{1}{2}\phi\,\Delta^{-1}\phi + \phi\circ J} = e^{S_I[\frac{d}{dJ}]}Z_0[J] \quad . \tag{3.13}$$

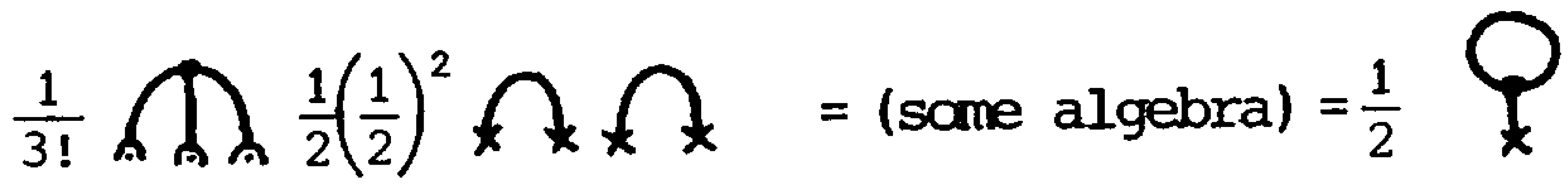
One expands both the interaction operator and the free field functional (3.7) as power series, and collects the nonvanishing terms:



where



For example,

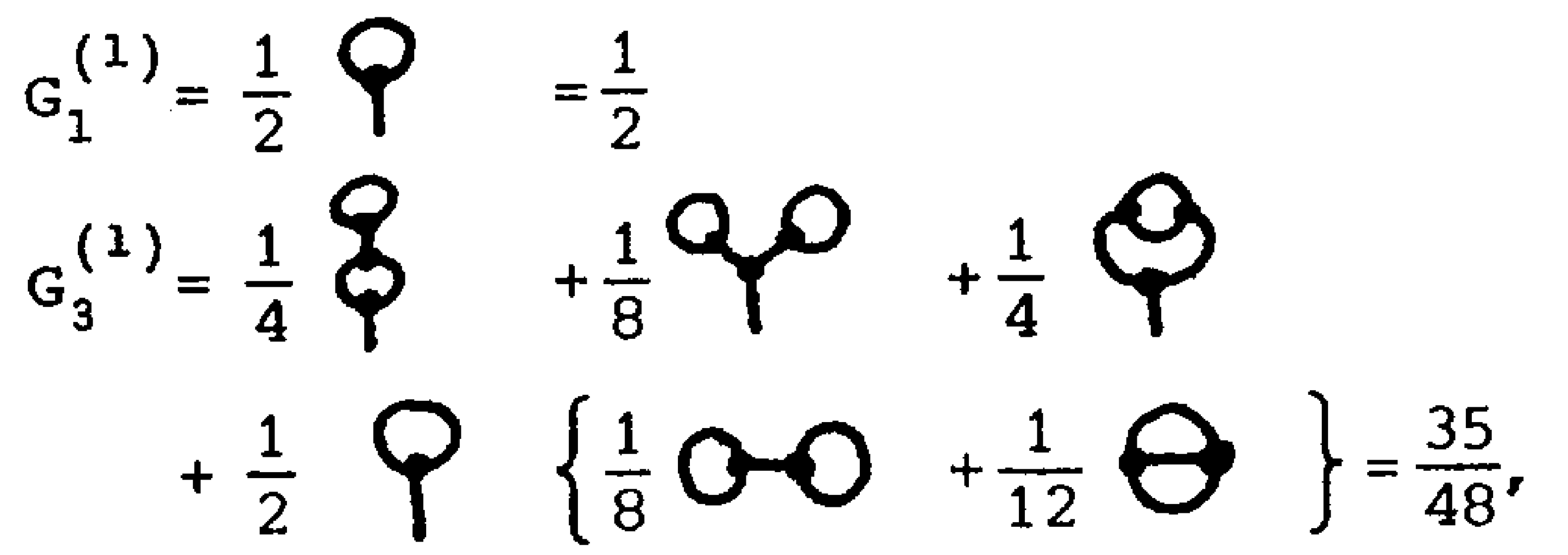


This is called the Wick expansion. It gives all the diagrams with the correct combinatoric factors, but is quite tedious. In practice, I prefer the DS equations.

Exercise 3.C.1 Use the Wick expansion (3.13) to show that for zero-dimensional ϕ^3 theory (exercise 2.E.3):

$$G_k^{(m)} = \frac{(3k+m-1)!!}{k!\,(3!)^k} \qquad \text{if } 3k+m \text{ even}$$
$$= 0 \qquad \text{otherwise}$$

For example,



Hint: use the combinatorial identity

$$\left.\frac{d^k}{dJ^k}\, e^{J^2/2}\right|_{J=0} = (k-1)!!, \quad k \text{ even} .$$

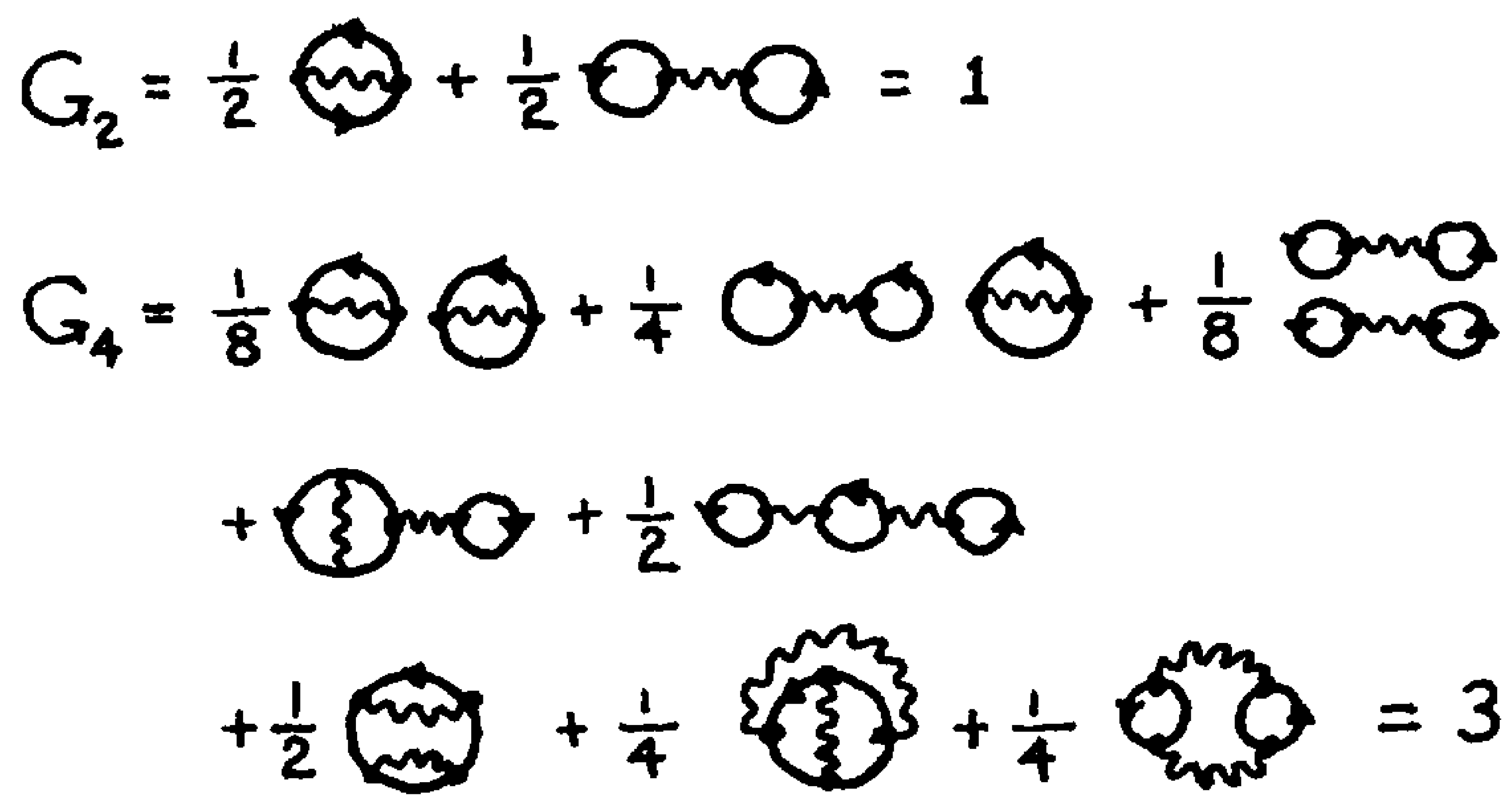
Exercise 3.C.2 Counting QED diagrams. Consider a zero-dimensional QED-like action

$$S = -\,\bar{\psi}\psi - \tfrac{1}{2}A^2 + g\bar{\psi}A\psi + \bar{\eta}\psi + \bar{\psi}\eta + JA .$$

Show by Wick expansion that

$$G_k^{(e,p)} = \frac{(k+e)!\,(k+p-1)!!}{k!} , \quad k+p \text{ even},$$

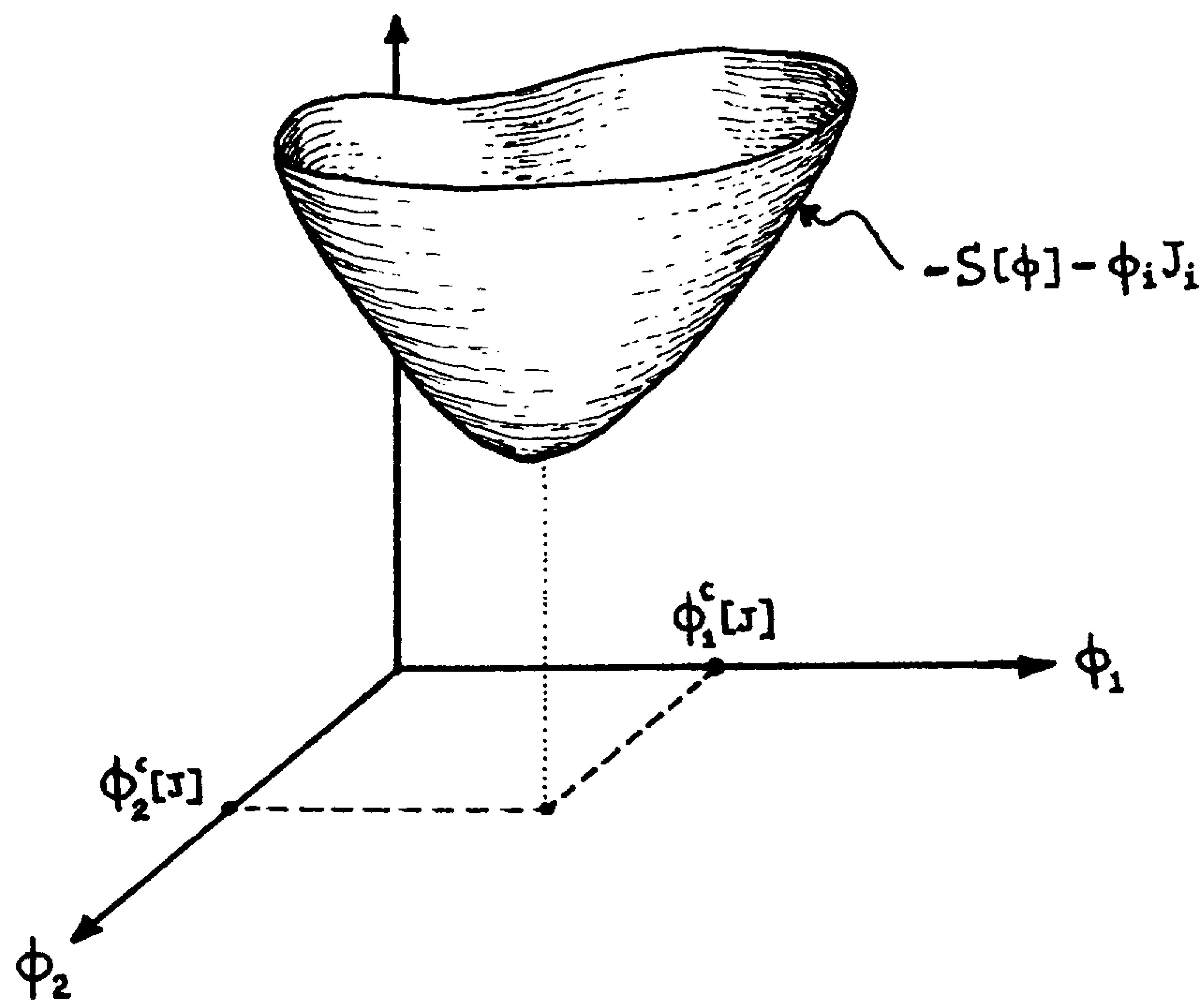
where e is the number of electron lines traversing the diagram, p is the number of photon legs, and k is the number of vertices. For example:



Hint: use $\left(\frac{d}{d\bar{\eta}}\frac{d}{d\eta}\right)^k e^{\eta\bar{\eta}} = k!$

D. Tree expansion

Let us take the path integral (3.9) very literally, and look at it as an ordinary multidimensional integral. We take ϕ_i to be real variables, and action a real function. The integral is finite only if the action is large and negative (high price of straying from the beaten path) almost everywhere, except for some localized regions of the ϕ-space. Highly idealized, the action looks something like this (we have suppressed an infinity of other coordinates):



(3.14)

The path integral will be dominated by the value of the action at the maximum (or maxima). ϕ^c, the location of the maximum, is determined by the extremum condition (cf.(3.9))

$$\frac{dS[\phi^c]}{d\phi_i} + J_i = 0 \ . \tag{3.15}$$

Hence the path integral is dominated by the solutions of the <u>classical equations of motion</u>. That these are really the familiar classical equations of motion can be seen by abandoning for a moment the collective index notation and writing out the integrations in the euclidean ϕ^3 action explicitly:

$$S[\phi] = -\int dx\left[\tfrac{1}{2}\phi(x)\,(-\partial^2 + m^2)\,\phi(x) + \frac{g}{3!}\phi^3(x)\right]$$

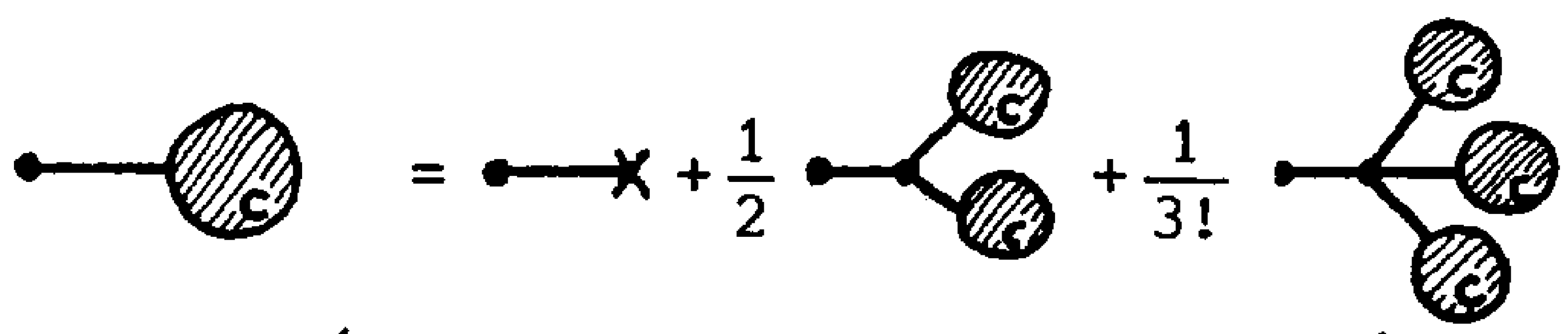
$$-\frac{\delta S[\phi]}{\delta\phi(x)} = (-\partial^2 + m^2)\,\phi(x) + \frac{g}{2}\phi^2(x) = J(x) \ . \tag{3.16}$$

The classical equations of motion differ from the quantum equations of motion (the DS equations (2.33)) by the absence of $d/d\phi$ terms. To interpret the classical solutions diagrammatically, we split the action into a quadratic part and an interaction part, as in (2.13):

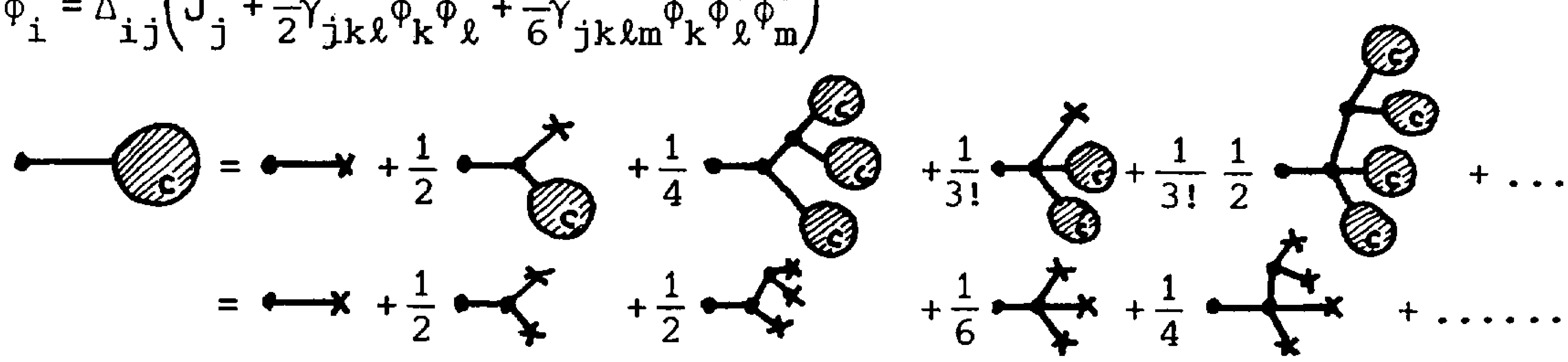
$$-\Delta^{-1}_{ij}\phi^c_j + \frac{dS_I[\phi^c]}{d\phi_i} + J_i = 0 \ ,$$

$$\phi^c_i[J] = \Delta_{ij}\left(J_j + \frac{dS_I[\phi^c]}{d\phi_j}\right) \ . \tag{3.17}$$

Unlike the quantum DS equations (2.21), the classical equations involve no loop terms. The iteration of the classical equations results in the <u>tree expansion</u>:



$$\phi^c_i = \Delta_{ij}\left(J_j + \frac{1}{2}\gamma_{jk\ell}\phi^c_k\phi^c_\ell + \frac{1}{6}\gamma_{jk\ell m}\phi^c_k\phi^c_\ell\phi^c_m\right)$$



(3.18)

This expression for the expectation value of a field is classical or deterministic in the sense that it involves no summations over virtual excitations, so it does not "feel" the probabilistic (quantum) aspects of the theory. It is also a way of getting at non-perturbative effects (such as spontaneous symmetry breaking): ϕ^c represents an infinite resummation which replaces the false vacuum ($\langle\phi\rangle \neq 0$) by the true ground state ($\langle\phi-\phi^c\rangle = 0$).

E. Legendre transformations

The classical approximation to a path integral is the value of the integrand at its extremum (3.15) (up to an irrelevant overall normalization factor):

$$Z_c[J] = e^{S[\phi^c]+\phi_i^c J_i} ,$$
$$W_c[J] = S[\phi^c] + \phi_i^c J_i \ . \tag{3.19}$$

The 1PI generating functional $\Gamma[\phi]$ satisfies extremum condition (2.27), analogous to the classical equations of motion (3.15). Indeed, the diagrammatic relation (2.24) between the connected and the 1PI Green function is a tree expansion of the connected Green functions, with all quantum loops confined to 1PI Green functions. Hence the 1PI generating functional $\Gamma[\phi]$ can be interpreted as an <u>effective</u> (or quantum) <u>action</u>, which satisfies the classical equations of motion (3.15), and where all quantum (or fluctuation) effects are incorporated into effective (proper) vertices, i.e. 1PI Green functions. Equation (3.19) becomes a relation between the connected and the 1PI Green functions:

$$W[J] = \Gamma[\phi] + \phi_i J_i \ . \tag{3.20}$$

This is just the Legendre transformation (2.28).

F. Saddlepoint expansion

The classical (tree, Born) approximation to Green functions is given by (3.19). The first quantum (or statistical fluctuation) correction is obtained by approximating the bottom of the potential (3.14) by a parabola, i.e. by keeping the quadratic term in the Taylor expansion

$$\overset{\curvearrowright\ = 0}{}$$

$$S[\phi] + \phi_i J_i = S[\phi^c] + \phi_i^c J_i + (\phi_i - \phi_i^c)\left(\frac{dS[\phi^c]}{d\phi_i} + J_i\right)$$
$$+ \frac{1}{2}(\phi_i - \phi_i^c)\frac{d^2S[\phi^c]}{d\phi_i d\phi_j}(\phi_j - \phi_j^c) + \frac{1}{3!} \cdots\cdot \tag{3.21}$$

The linear term vanishes because we are expanding around an extremum, and the quadratic term can be integrated by the Gaussian integration (3.6):

$$Z[J] \simeq e^{S[\phi^c] + \phi_i^c J_i} \frac{1}{\sqrt{\mathrm{Det}\left(-\frac{d^2S[\phi^c]}{d\phi_i d\phi_j}\right)}} \ . \tag{3.22}$$

To interpret the determinant diagrammatically, we use

$$\mathrm{Det}\, M = e^{\mathrm{tr}\, \ell n M} \ . \tag{3.23}$$

Derivation:

$$\delta(\ell n\, \mathrm{Det}\, M) = \ell n\, \mathrm{Det}(M + \delta M) - \ell n\, \mathrm{Det}\, M$$
$$= \ell n\, \mathrm{Det}(1 + \delta M/M)$$

$$\mathrm{Det}(1+\Delta) = (1+\Delta_{11})(1+\Delta_{22})\cdots\cdot$$
$$- \Delta_{21}\Delta_{12}(1+\Delta_{33})\cdots\cdot + \cdots$$
$$= 1 + \mathrm{tr}\Delta + O(\Delta^2)$$

$$\simeq \ell n(1 + \mathrm{tr}\,\delta M/M)$$
$$\simeq \mathrm{tr}\,\delta M/M$$
$$= \mathrm{tr}\,\delta(\ell n M)$$
$$= \delta(\mathrm{tr}\, \ell n\, M)$$

hence

$$\ell n\, \mathrm{Det}\, M = \mathrm{tr}\, \ell n\, M \ .$$

This is obvious for diagonalizable matrices:

$$\mathrm{Det}\, M = \prod_i \lambda_i = e^{\sum_i \ell n \lambda_i} = e^{\mathrm{tr}\, \ell n\, M} \ . \qquad \text{QED}$$

Splitting S″ into the bare propagator and the interactions with the classical background field

$$\frac{d^2S[\phi^c]}{d\phi_i d\phi_j} = -\Delta_{ij}^{-1} + \gamma_{ij}[\phi^c] \ ,$$

$\gamma_{ij}[\phi^c] =$  $+ \cdots\cdots$ (3.24)

we can write the first approximation to Z[J] as

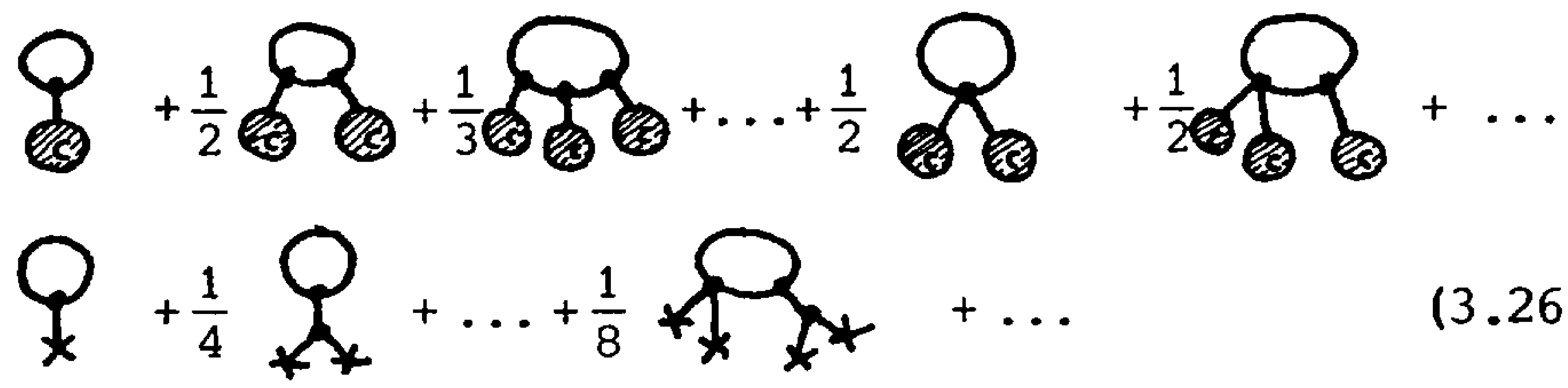
$$Z[J] \simeq e^{S[\phi^c] - \frac{1}{2}\mathrm{tr}\,\ln(1-\Delta\gamma[\phi^c]) + \phi_i^c J_i} \cdot \sqrt{\mathrm{Det}\Delta} \quad ,$$

(the overall $\sqrt{\mathrm{Det}\Delta}$ factor can be dropped). In this approximation the effective action is given by

$$\Gamma[\phi^c] = S[\phi^c] + \frac{1}{2}\sum_{k=1}^{\infty}\frac{1}{k}\,\mathrm{tr}(\Delta\gamma[\phi^c])^k \ . \tag{3.25}$$

This is called the one-loop effective action, as its diagrammatic expansion consists of all one-loop diagrams:

$$\Gamma^{(1)}[\phi^c] = \frac{1}{2}\Big\{$$



(3.26)

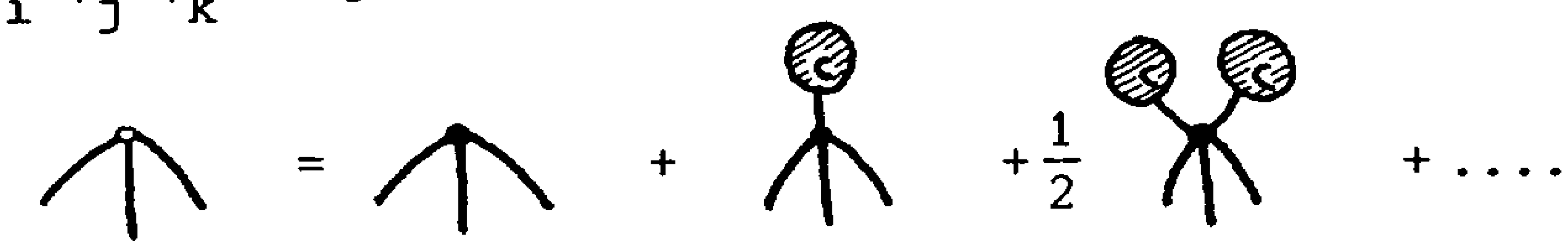
The higher loop contributions to the effective action can be computed by the ordinary perturbation expansion, with ϕ^c playing the role of a "background field", i.e. the field which describes the classical background configuration in which the propagation and the interactions take place. This expansion is carried out in the next exercise.

Exercise 3.F.1 The loop expansion for effective action. Introduce an auxiliary source K_i in the saddlepoint expansion (3.21)

$$Z[J] = e^{S[\phi^c]+\phi_i^c J_i}\int[d\phi]\,e^{\frac{1}{2}\phi_i\frac{d^2S[\phi^c]}{d\phi_i d\phi_j}\phi_j + S_I^c[\phi] + \phi_i K_i}\Big|_{K=0}$$

$$S_I^c[\phi] = \frac{1}{3!}\frac{d^3S[\phi^c]}{d\phi_i d\phi_j d\phi_k}\phi_k\phi_j\phi_i + \frac{1}{4!}\frac{d^4S[\phi^c]}{d\ldots\ldots}\ \ldots\ldots$$

$$\frac{d^3S[\phi^c]}{d\phi_i d\phi_j d\phi_k} =$$



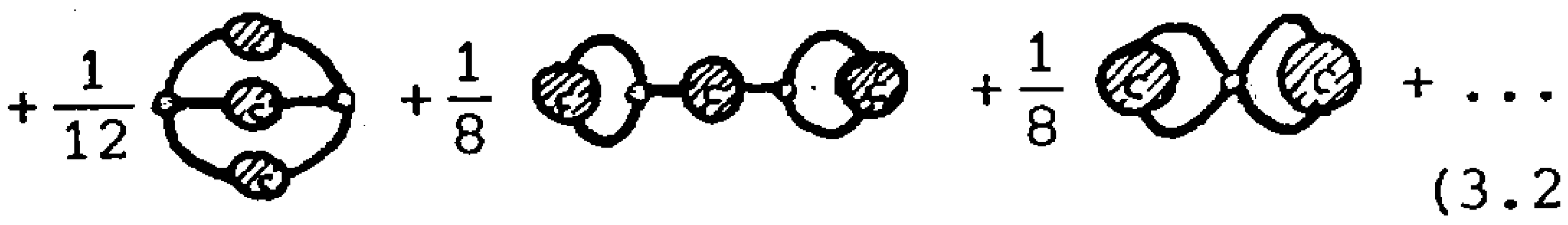
Now we can use the Wick expansion (3.13) to write the loop expansion for J:

$$Z[J] = e^{S[\phi^c]+\phi_i^c J_i + \frac{1}{2}\,\mathrm{tr}\,\ln\left(-\frac{d^2S[\phi_c]}{d\phi_i d\phi_j}\right)}$$

$$\times\, e^{S_I^c[\frac{d}{dK}]}\, e^{\frac{1}{2}K_i\left(-\frac{1}{\frac{d^2S[\phi^c]}{d\phi_i d\phi_j}}\right)K_j}\Bigg|_{K=0} \ .$$

We can interpret this expansion as an ordinary perturbation expansion for vacuum bubbles, with propagators and vertices dependent on the classical background field ϕ^c. All possible insertions of sources J_i are summed up into tree insertions by $\phi^c[J]$. Compute the beginning of this expansion

$$W[J] = S[\phi^c] + \tfrac{1}{2}\,\mathrm{tr}\,\ln(\Delta\gamma[\phi^c])$$



(3.27)

Compare with the results of exercise 2.I.2. Write down the beginning of the loop expansion for the effective action $\Gamma[\phi]$.

Exercise 3.F.2 Consider a QED-like theory from exercise 2.D.1. The path integral can be written as

$$Z[J,\bar\eta,\eta] = \int [dA]\, e^{-\frac{1}{2}A_\mu D^{-1}_{\mu\nu}A_\nu + J_\mu A_\mu}\, Z[\bar\eta,\eta]_A$$

$$Z[\bar\eta,\eta]_A = \int [d\bar\psi][d\psi]\, e^{-\bar\psi(\Delta^{-1} - A\!\!\!/)\psi + \bar\eta\psi + \bar\psi\eta}$$

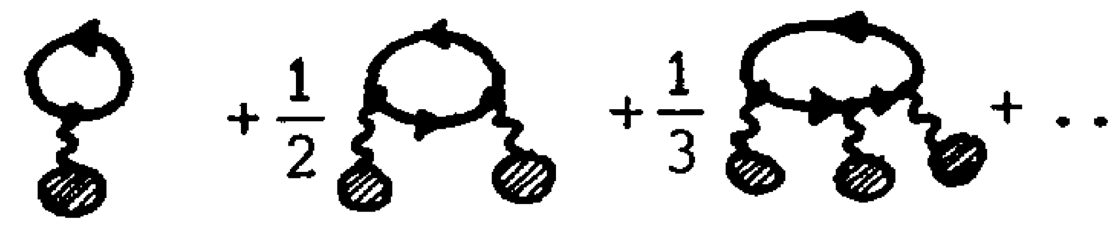
$$A\!\!\!/_{\alpha\beta} = A_\mu (\gamma^\mu)_{\alpha\beta}\ .$$

$Z[\bar\eta,\eta]_A$ can be interpreted as the generating functional for the free electrons propagating in the background field $A\!\!\!/_{\alpha\beta}$. Show that

$$Z[\bar\eta,\eta]_A = e^{-\ln \mathrm{tr}(1-\Delta A\!\!\!/) + \bar\eta\Delta\frac{1}{1-A\!\!\!/\Delta}\eta}\ . \qquad (3.28)$$

The trace part accounts for all virtual electron loops:

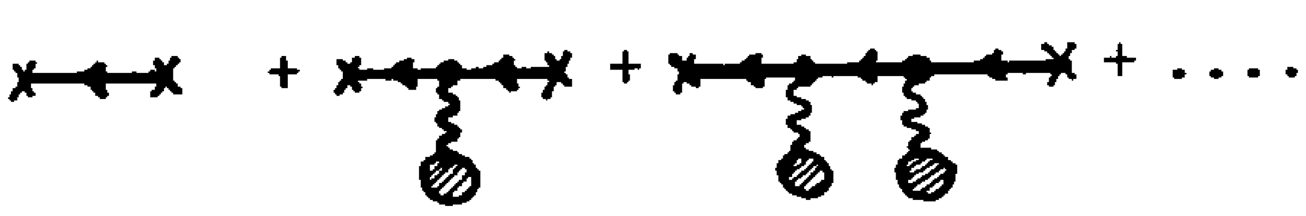
$$-\ln \mathrm{tr}(1-\Delta A\!\!\!/) =$$



(3.29)

while the source term describes the propagation of the electron in the background A_μ field:

$$\bar\eta\Delta\frac{1}{1-A\!\!\!/\Delta}\eta =$$



(3.30)

Exercise 3.F.3 Counting QED diagrams. (Continuation of exercise 3.C.2). Integrate over "photon" fields to obtain

$$Z[J,\bar\eta,\eta] = e^{-\ln(1-g\frac{d}{dJ}) + \bar\eta\frac{1}{1-g\frac{d}{dJ}}\eta}\, e^{J^2/2}\ .$$

Show that the number of full electron propagator diagrams without electron loops is

$$D_k = (k-1)!!\ , \qquad k \text{ even}$$

$D_2 =$ 

$D_4 =$  , etc.

What is the number of the photon self-energy graphs with only one electron loop? Furry's theorem says that all diagrams with electron loops with odd numbers of photon legs vanish. They can be eliminated from the loop expansion by replacement

$$\ell n(1-gA) \to \tfrac{1}{2}\ell n(1-gA) + \tfrac{1}{2}\ell n(1+gA) \quad .$$

Show that the number of full electron propagators (electron loops included) is

$$D_k = \frac{(k+1)!!\,(k-1)!!}{k!!}, \quad k \text{ even} \quad .$$

Check that $D_4 = \frac{45}{8}$ (this is not an integer, as disconnected graphs like

$\frac{1}{4}$ 

are included).

G. Point transformations

One of the main advantages of the path integral formalism is the compactness of Ward identities. The key idea is simple. In

$$Z[J] = \int [d\phi] e^{S[\phi] + J_i \phi_i}$$

the left-hand side is independent of ϕ, hence invariant under infinitesimal point transformations

$$\phi_i \to \phi_i + \varepsilon F_i[\phi]$$

$$F_i[\phi] = f_i + f_{ij}\phi_j + f_{ijk}\phi_j\phi_k + \ldots\ldots \quad . \tag{3.31}$$

The Jacobian for this change of variables is (dropping terms of order ε^2 and higher):

$$[d\phi] \to [d\phi]\det\left|\delta_{ij} - \varepsilon\frac{dF_i[\phi]}{d\phi_j}\right| = [d\phi]\left(1 - \varepsilon\frac{dF_i[\phi]}{d\phi_i}\right) \ .$$

Collecting all terms up to order ε we obtain

$$Z[J] = \int [d\phi]\left(1 - \varepsilon\frac{dF_i[\phi]}{d\phi_i}\right) e^{S[\phi]+J_i\phi_i}\left(1 + \varepsilon\left(\frac{dS[\phi]}{d\phi_i} + J_i\right)F_i[\phi]\right) ,$$

$$0 = \int [d\phi]\left\{\left(\frac{dS[\phi]}{d\phi_i} + J_i\right)F_i[\phi] - \frac{dF_i[\phi]}{d\phi_i}\right\} e^{S[\phi]+J_i\phi_i} \quad .$$

Remembering the equivalence $\phi_i \leftrightarrow d/dJ_i$ we can write this as

$$\left\{\left(\frac{dS}{d\phi_i}\left[\frac{d}{dJ}\right] + J_i\right)F_i\left[\frac{d}{dJ}\right] - \frac{dF_i}{d\phi_i}\left[\frac{d}{dJ}\right]\right\} Z[J] = 0 \quad . \tag{3.32}$$

We have already, unknowingly, used a special case of this identity. If $F_i[\phi] = f_i =$ constant (a translation), then

$$\left(\frac{dS}{d\phi_i}\left[\frac{d}{dJ}\right] + J_i\right)Z[J] = 0 \quad . \tag{2.15}$$

The Dyson-Schwinger equations are consequences of the translational invariance of path integrals. A more interesting situation arises if (3.31) is a symmetry of the action

$$\frac{dS[\phi]}{d\phi_i} F_i[\phi] = 0 \quad . \tag{3.33}$$

If this transformation also leaves invariant the measure $[d\phi]$, then (3.32) reduces to a <u>Ward identity</u>:

$$J_i F_i\left[\frac{d}{dJ}\right] Z[J] = 0 \ . \tag{3.34}$$

The Ward identities are immensely important. They tell us how the symmetries of the action (classical theory) relate various Green functions (quantum theory). About this - later.

<u>Exercise 3.G.1</u> <u>Derivative interactions</u>. Throughout these notes we treat the sums over discrete indices and the integrals over continuous variables as the one and the same thing. However, for derivative interactions we must be more careful. Consider a one-dimensional example with action

$$S = \int dt\ \mathcal{L}(t)$$

where the Lagrangian density includes derivatives:

$$\mathcal{L}(t) = \tfrac{1}{2}\dot{\phi}_i K_{ij}\dot{\phi}_j + L_i\dot{\phi}_i - V(\phi) .$$

Show that the correct definition of the path integral is

$$Z[J] = \int [d\phi]\,(\mathrm{Det}\,K)^{\frac{1}{2}}\, e^{S+\int dt J_i\phi_i} \ .$$

Hint: the path integral must be invariant under variable change

$$K \to \frac{\partial\phi}{\partial\bar{\phi}}\,\bar{K}\,\frac{\partial\phi}{\partial\bar{\phi}}, \quad [d\phi] \to [d\bar{\phi}]\mathrm{Det}\left(\frac{\partial\phi}{\partial\bar{\phi}}\right) .$$

H. Summary

The basic assumption of the statistical (quantum) mechanics is that the physical processes can be described additively, as sums of probabilities (amplitudes). Whether we describe these sums by diagrams (generating functional formalism) or field configurations (path integral formalism) is largely a matter of convenience. The two formalisms offer two ways of visualising the relation between the classical and the quantum physics.

In the path integral picture, the transition rates are dominated by the valleys of the potential, and the quantum effects are the heavily penalized forays up the hillsides. In the statistical mechanics they are suppressed by small Boltzmann weights; in quantum mechanics they are suppressed by destructive interference of phases.

In the Feynman diagram picture, physical processes are dominated by classical propagation (tree diagrams) and the quantum effects are represented by internal loops (virtual excitations).

The two pictures are related by

$$G_{ij..k} = \langle \phi_i \phi_j .. \phi_k \rangle = \frac{1}{Z[0]} \int [d\phi] \phi_i \phi_j .. \phi_k e^{S[\phi]} \ .$$

A path integral is dominated by the extremal solutions of the classical equations of motion

$$\frac{dS[\phi^c]}{d\phi_i} + J_i = 0 \ .$$

The quantum effects can be included systematically by the loop expansion of the effective (quantum) action:

$$\Gamma[\phi^c] = S[\phi^c] - \tfrac{1}{2} \mathrm{tr}\, \ln (1 - \Delta\gamma[\phi^c]) + \ldots.$$

$$\gamma_{ij}[\phi^c] = \frac{d^2 S_I[\phi^c]}{d\phi_i d\phi_j} \ .$$

The classical symmetries of the action

$$F_i[\phi] \frac{dS[\phi]}{d\phi_i} = 0$$

imply the quantum symmetries, or Ward identities

$$J_i F_i \left[\frac{d}{dJ} \right] Z[J] = 0 \ .$$